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Adrián GONZÁLEZ-PÉREZ & Marius JUNGE & Javier PARCET

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

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Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : `smf@smf.univ-mrs.fr`

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SMOOTH FOURIER MULTIPLIERS IN GROUP ALGEBRAS VIA SOBOLEV DIMENSION

BY ADRIÁN GONZÁLEZ-PÉREZ, MARIUS JUNGE AND JAVIER PARCET

ABSTRACT. – We investigate Fourier multipliers with smooth symbols defined over locally compact Hausdorff groups. Our main results in this paper establish new Hörmander-Mikhlin criteria for spectral and non-spectral multipliers. The key novelties which shape our approach are three. First, we control a broad class of Fourier multipliers by certain maximal operators in noncommutative L_p spaces. This general principle—exploited in Euclidean harmonic analysis during the last 40 years—is of independent interest and might admit further applications. Second, we replace the formerly used cocycle dimension by the Sobolev dimension. This is based on a noncommutative form of the Sobolev embedding theory for Markov semigroups initiated by Varopoulos, and yields more flexibility to measure the smoothness of the symbol. Third, we introduce a dual notion of polynomial growth to further exploit our maximal principle for non-spectral Fourier multipliers. The combination of these ingredients yields new L_p estimates for smooth Fourier multipliers in group algebras.

RÉSUMÉ. – Nous étudions des multiplicateurs de Fourier à symboles réguliers sur des groupes localement compacts. De nouveaux critères de Hörmander-Mikhlin pour des multiplicateurs spectraux et non spectraux sont établis. Notre approche se base sur trois nouveaux résultats clés. Premièrement, nous utilisons certains opérateurs maximaux dans des espaces L_p non commutatifs pour obtenir un contrôle sur de larges classes de multiplicateurs. Ce principe général — exploité en analyse harmonique euclidienne ces 40 dernières années — présente un intérêt indépendant et pourrait admettre de nouvelles applications. Deuxièmement, en établissant une version non commutative de la théorie de plongement de Sobolev pour les semigroupes de Markov initiée par Varopoulos, la dimension de cocycle utilisée auparavant est remplacée par la dimension de Sobolev. Ceci permet plus de flexibilité sur la régularité du symbole. Enfin, nous introduisons une notion duale de la croissance polynomiale pour exploiter davantage notre principe du maximum sur des multiplicateurs de Fourier non spectraux. La combinaison de ces ingrédients produit de nouvelles estimations L_p pour des multiplicateurs de Fourier réguliers dans des algèbres de groupe.

Introduction

The aim of this paper is to study Fourier multipliers on group von Neumann algebras for locally compact Hausdorff groups. More precisely, the relation between the smoothness of

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their symbols and L_p-boundedness. This is a central topic in Euclidean harmonic analysis. In the context of nilpotent groups, it has also been intensively studied in the works of Cowling, Müller, Ricci, Stein and others. In this paper we will consider the dual problem, placing our nonabelian groups in the frequency side. Today it is well understood that the dual of a nonabelian group can only be described as a quantum group, its underlying algebra being the group von Neumann algebra. The interest of Fourier multipliers over such group algebras was recognized early in the pioneering work of Haagerup [11], as well as in the research carried out thereafter. It was made clear how Fourier multipliers on these algebras can help in their classification, through the use of certain approximation properties which become invariants of the algebra. Unfortunately, the literature on this topic does not involve the L_p-theory—with a few exemptions like [21] and the very recent paper of Lafforgue and de la Salle [25]—as it is mandatory from a harmonic analysis viewpoint. In this respect, our work is a continuation of [18, 19] where 1-cocycles into finite-dimensional Hilbert spaces were used to lift multipliers from the group into a more Euclidean space. This yields Hörmander-Mikhlin type results depending of the dimension of the Hilbert space involved. Here, we shall follow a different approach through the introduction of new notions of dimension allowing more room for the admissible class of multipliers. These notions rely on noncommutative forms of the Sobolev embedding theory for Markov semigroups, which carrie an 'encoded geometry' in the commutative setting. Prior to that, we need to investigate new maximal bounds whose Euclidean analogs are central in harmonic analysis. In this paper we shall limit ourselves to unimodular groups to avoid technical issues concerning modular theory.

This text is divided into three blocks which are respectively devoted to maximal bounds, Sobolev dimension and polynomial co-growth. Let us first put in context our maximal estimates for Fourier multipliers. Given a symbol $m : \mathbb{R}^n \to \mathbb{C}$ with corresponding Fourier multiplier T_m, there is a long tradition in identifying maximal operators $\mathcal{M}$ which satisfy the weighted L_2-norm inequality below for all admissible input functions f and weights w

$$(\mathrm{W}L_2) \qquad \int_{\mathbb{R}^n} |T_m f|^2 w \lesssim \int_{\mathbb{R}^n} |f|^2 \mathcal{M} w.$$

It goes back to the work of Córdoba and Fefferman in the 70's. This general principle has deep connections with Bochner-Riesz multipliers and also with A_p weight theory. The Introduction of [2] gives a very nice historical summary and new results in this direction. The main purpose of this estimate is that elementary duality arguments yield for $p > 2$ that

$$\big\| T_m : L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n) \big\| \lesssim \big\| \mathcal{M} : L_{(p/2)'}(\mathbb{R}^n) \to L_{(p/2)'}(\mathbb{R}^n) \big\|^{\frac{1}{2}}.$$

The most general noncommutative form of this inequality would require too much terminology for this Introduction. Instead, let us just introduce the basic concepts to give a reasonable but weaker statement. Stronger results will be given in the body of the paper. Let G be a locally compact Hausdorff group. If we write μ for the left Haar measure of G and λ for the left regular representation $\lambda : G \to \mathcal{B}(L_2 G)$, the group von Neumann algebra $\mathcal{L}G$ is the weak operator closure in $\mathcal{B}(L_2 G)$ of $\lambda(L_1(G))$. We refer to Section 1 for a construction of the Plancherel weight τ on $\mathcal{L}G$, a noncommutative substitute of the Haar measure. Note that τ is tracial iff G is unimodular—which we assume—and it coincides with the finite trace given by $\tau(x) = \langle \delta_e, x\delta_e \rangle$ when G is discrete. In the unimodular case, $(\mathcal{L}G, \tau)$ is a semifinite von Neumann algebra with a trace and it is easier to construct the noncommutative

L_p-spaces $L_p(\mathcal{L}G, \tau)$ with norm $\|x\|_p = \tau(|x|^p)^{1/p}$, where $|x|^p = (x^*x)^{p/2}$ by functional calculus on the (unbounded) operator x^*x. Given a bounded symbol $m : G \to \mathbb{C}$, the corresponding Fourier multiplier is densely defined by $T_m\lambda(f) = \lambda(mf)$. Alternatively, it will be useful to understand these operators as convolution maps in the following way

$$T_m(x) = \lambda(m) \star x = (\tau \otimes \mathrm{Id})\big(\delta\lambda(m)\,(\sigma x \otimes \mathbf{1})\big),$$

where $\delta : \mathcal{L}G \to \mathcal{L}G \,\overline{\otimes}\, \mathcal{L}G$ is determined by $\delta(\lambda_g) = \lambda_g \otimes \lambda_g$ and $\sigma : \mathcal{L}G \to \mathcal{L}G$ is the anti-automorphism given by linear extension of $\sigma(\lambda_g) = \lambda_{g^{-1}}$. The first map is called the comultiplication map for $\mathcal{L}G$, whereas σ is the corresponding coinvolution. Our next ingredient is the L_p-norm of maximal operators. Given a family of noncommuting operators $(x_\omega)_\omega$ affiliated to a semifinite von Neumann algebra $\mathcal{M}$, their supremum is not well-defined. We may however consider their L_p-norms through

$$\Big\| {\sup_{\omega\in\Omega}}^+ x_\omega \Big\|_{L_p(\mathcal{M})} = \big\| (x_\omega)_{\omega\in\Omega} \big\|_{L_p(\mathcal{M}; L_\infty(\Omega))},$$

where the mixed-norm $L_p(L_\infty)$-space has a nontrivial definition obtained by Pisier for hyperfinite $\mathcal{M}$ in [31] and later generalized in [15, 20]. This definition recovers the norm in $L_p(\Sigma; L_\infty(\Omega))$ for abelian $\mathcal{M} = L_\infty(\Sigma)$, further details in Section 1. Finally, conditionally negative lengths $\psi : G \to \mathbb{R}_+$ are symmetric functions vanishing at the identity e which satisfy $\sum_{g,h} \overline{a_g} a_h \psi(g^{-1}h) \le 0$ for any family of coefficients with $\sum_g a_g = 0$. Due to its one-to-one relation to Markov convolution semigroups, they will play a crucial role in this paper. In the classical multiplier theorems, the symbols m are cut out with functions $\eta(|\xi|)$ for some compactly supported $\eta \in C^\infty(\mathbb{R}_+)$. Our techniques do not allow us to use compactly supported functions in $\mathbb{R}_+$. Instead, we are going to use analytic functions decaying fast near 0 and near ∞. We will call such η an $\mathcal{H}_0^\infty$-cut-off, see Section 1 for the precise definitions. The archetype of such functions will be $\eta(z) = z\,e^{-z}$. When such function is real-valued for every $z \in \mathbb{R}_+$ we will say that η is real. We will denote by $L_p^\circ(\mathcal{L}G)$ the noncommutative L_p-space $L_p(\mathcal{L}G)$ modulo the functions supported in $G_0 = \{g \in G : \psi(g) = 0\}$, see Section 1.1.7 for details.

Theorem A. – *Let G be a unimodular group equipped with any conditionally negative length $\psi : G \to \mathbb{R}_+$. Let η be a real $\mathcal{H}_0^\infty$-cut-off and $m : G \to \mathbb{R}$ an essentially bounded symbol satisfying that $m(g^{-1}) = m(g)$ for every $g \in G$. Assume $B_t = \lambda(m\eta(t\psi))$ admits a decomposition $B_t = \Sigma_t M_t$ with M_t positive and satisfying $M_t = \sigma M_t$, and consider the convolution map $\mathcal{R}(x) = (|M_t|^2 \star x)_{t\ge 0}$. Then the following inequality holds for $2 < p < \infty$*

$$\|T_m\|_{\mathcal{B}(L_p^\circ(\mathcal{L}G))} \lesssim_{(p)} \Big(\sup_{t\ge 0} \|\Sigma_t\|_2\Big) \Big\| \mathcal{R} : L_{(p/2)'}(\mathcal{L}G) \to L_{(p/2)'}(\mathcal{L}G; L_\infty) \Big\|^{\frac12},$$

By duality, a similar stamens holds for $1 < p < 2$. Moreover, the assumptions of η being real, of m satisfying that $m(g^{-1}) = m(g)$ and of M_t being positive and satisfying that $M_t = \sigma M_t$ can be removed if we allow slight modifications in the statement of the theorem, like framing the conclusion in terms of noncommutative Hardy spaces. Theorem A combines in a very neatly way noncommutative generalizations of (WL_2) with square function estimates. In the particular case of Hörmander-Mikhlin symbols—as we shall see along this paper—the decomposition splits the assumptions. Namely, the L_2-norm of Σ_t is bounded using the smoothness condition while the maximal $\mathcal{R}$ is bounded through the geometrical