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POISSON BOUNDARIES OF MONOIDAL CATEGORIES

BY SERGEY NESHVEYEV AND MAKOTO YAMASHITA

ABSTRACT. – Given a rigid C*-tensor category $\mathcal{C}$ with simple unit and a probability measure μ on the set of isomorphism classes of its simple objects, we define the Poisson boundary of $(\mathcal{C}, \mu)$. This is a new C*-tensor category $\mathcal{P}$, generally with nonsimple unit, together with a unitary tensor functor $\Pi\colon \mathcal{C} \to \mathcal{P}$. Our main result is that if $\mathcal{P}$ has simple unit (which is a condition on some classical random walk), then Π is a universal unitary tensor functor defining the amenable dimension function on $\mathcal{C}$. Corollaries of this theorem unify various results in the literature on amenability of C*-tensor categories, quantum groups, and subfactors.

RÉSUMÉ. – Etant données une C*-catégorie tensorielle rigide $\mathcal{C}$ dont l'objet unité est simple ainsi qu'une mesure de probabilité μ sur l'ensemble de classes d'isomorphisme des objets simples, nous définissons la frontière de Poisson de $(\mathcal{C}, \mu)$. C'est une nouvelle C*-catégorie tensorielle $\mathcal{P}$ dont l'objet unité n'est pas, en général, simple, couplée avec un foncteur unitaire tensoriel $\Pi\colon \mathcal{C} \to \mathcal{P}$. Notre résultat principal assure que si l'objet unité de $\mathcal{P}$ est simple (ce qui se traduit par une condition sur une certaine marche aléatoire classique), alors Π est un foncteur unitaire tensoriel universel qui définit la fonction de dimension moyennable sur $\mathcal{C}$. Les corollaires de ce théorème unifient différents résultats connus sur la moyennabilité des C*-catégories tensorielles, des groupes quantiques et des sous-facteurs.

Introduction

The notion of amenability for monoidal categories first appeared in Popa's seminal work [29] on classification of subfactors as a crucial condition defining a class of inclusions admitting good classification. He then gave various characterizations of this property analogous to the usual amenability conditions for discrete groups: a Kesten type condition on the norm of the principal graph, a Følner type condition on the existence of almost invariant sets, and a Shannon-McMillan-Breiman type condition on relative entropy, to name a few.

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This stimulated a number of interesting developments in related fields of operator algebras. First, Longo and Roberts [19] developed a general theory of dimension for C*-tensor categories, and indicated that the language of sectors/subfactors is well suited for studying amenability in this context. Then Hiai and Izumi [12] studied amenability for fusion algebras/hypergroups endowed with a probability measure, and obtained many characterizations of this property in terms of random walks and almost invariant vectors in the associated ℓ^p-spaces. These studies were followed by the work of Hayashi and Yamagami [9], who established a way to realize amenable monoidal categories as bimodule categories over the hyperfinite II_1 factor.

In addition to subfactor theory, another source of interesting monoidal categories is the theory of quantum groups. In this framework, the amenability question concerns the existence of almost invariant vectors and invariant means for a discrete quantum group, or some property of the dimension function on the category of unitary representations of a compact quantum group [1, 32, 2]. Here, one should be aware that there are two different notions of amenability involved. One is coamenability of compact quantum groups (equivalently, amenability of their discrete duals) considered in the regular representations, the other is amenability of representation categories. These notions coincide only for quantum groups of Kac type.

In yet another direction, Izumi [13] developed a theory of noncommutative Poisson boundaries for discrete quantum groups in order to study the minimality (or lack thereof) of infinite tensor product type actions of compact quantum groups. From the subsequent work [16, 33] it became increasingly clear that for coamenable compact quantum groups the Poisson boundary captures a very elaborate difference between the two amenability conditions. Later, an important result on noncommutative Poisson boundaries was obtained by De Rijdt and Vander Vennet [6], who found a way to compute the boundaries through monoidal equivalences. In light of the categorical duality for compact quantum group actions recently developed in [5, 21], this result suggests that the Poisson boundary should really be an intrinsic notion of the representation category $\operatorname{Rep} G$ itself, rather than of the choice of a fiber functor giving a concrete realization of $\operatorname{Rep} G$ as a category of Hilbert spaces. Starting from this observation, in this paper we define Poisson boundaries for monoidal categories.

To be more precise, our construction takes a rigid C*-tensor category $\mathcal{C}$ with simple unit and a probability measure μ on the set $\operatorname{Irr}(\mathcal{C})$ of isomorphism classes of simple objects, and gives another C*-tensor category $\mathcal{P}$ together with a unitary tensor functor $\Pi\colon \mathcal{C} \to \mathcal{P}$. Although the category $\mathcal{P}$ is defined purely categorically, there are several equivalent ways to describe it, or at least its morphism sets, that are more familiar to the operator algebraists. One is an analog of the standard description of classical Poisson boundaries as ergodic components of the time shift. Another is in terms of relative commutants of von Neumann algebras, in the spirit of [19, 9, 13]. For categories arising from subfactors and quantum groups, this can be made even more concrete. For subfactors, computing the Poisson boundary essentially corresponds to passing to the standard model of a subfactor [29]. For quantum groups, not surprisingly as this was our initial motivation, the Poisson boundary of the representation category of G can be described in terms of the Poisson boundary of $\hat{G}$. The last result will be discussed in detail in a separate publication [25], since we also want to describe the action of $\hat{G}$ on the boundary in categorical terms and this would lead us away from the main subject of this paper.

Our main result is that if $\mathcal{P}$ has simple unit, which corresponds to ergodicity of the classical random walk defined by μ on $\mathrm{Irr}(\mathcal{C})$, then $\Pi\colon \mathcal{C} \to \mathcal{P}$ is a universal unitary tensor functor which induces the amenable dimension function on $\mathcal{C}$. From this we conclude that $\mathcal{C}$ is amenable if and only if there exists a measure μ such that Π is a monoidal equivalence. The last result is a direct generalization of the famous characterization of amenability of discrete groups in terms of their Poisson boundaries due to Furstenberg [7], Kaimanovich and Vershik [17], and Rosenblatt [31]. From this comparison it should be clear that, contrary to the usual considerations in subfactor theory, it is not enough to work only with finitely supported measures, since there are amenable groups which do not admit any finitely supported ergodic measures [17]. The characterization of amenability in terms of Poisson boundaries generalizes several results in [29, 19, 9]. Our main result also allows us to describe functors that factor through Π in terms of categorical invariant means. For quantum groups this essentially reduces to the equivalence between coamenability of G and amenability of $\hat{G}$ [32, 2].

Although our theory gives a satisfactory unification of various amenability results, the main remarkable property of the functor $\Pi\colon \mathcal{C} \to \mathcal{P}$ is, in our opinion, the universality. If the category $\mathcal{P}$ happens to have a simpler structure compared to $\mathcal{C}$, this universality allows one to reduce classification of functors from $\mathcal{C}$ inducing the amenable dimension function to an easier classification problem for functors from $\mathcal{P}$. This idea will be used in [26] to classify a class of compact quantum groups.

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1. Preliminaries

1.1. Monoidal categories

In this paper we study *rigid C*-tensor categories*. By now there are many texts covering the basics of this subject, see for example [35, 20, 24] and references therein. We mainly follow the conventions of [24], but for the convenience of the reader we summarize the basic definitions and facts below.

A *C*-category* is a category $\mathcal{C}$ whose morphism sets $\mathcal{C}(U, V)$ are complex Banach spaces endowed with complex conjugate involution $\mathcal{C}(U, V) \to \mathcal{C}(V, U)$, $T \mapsto T^*$ satisfying the C*-identity. Unless said otherwise, we always assume that $\mathcal{C}$ is closed under finite direct sums and subobjects. The latter means that any idempotent in the endomorphism ring $\mathcal{C}(X) = \mathcal{C}(X, X)$ comes from a direct summand of X.

A C*-category is said to be *semisimple* if any object is isomorphic to a direct sum of simple (that is, with the endomorphism ring $\mathbb{C}$) objects. We then denote the isomorphism classes of simple objects by $\mathrm{Irr}(\mathcal{C})$ and assume that this set is at most countable. Many results admit formulations which do not require this assumption and can be proved by considering subcategories generated by countable sets of simple objects, but we leave this matter to the interested reader.