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SYMBOLIC DYNAMICS FOR NON-UNIFORMLY HYPERBOLIC SURFACE MAPS WITH DISCONTINUITIES

BY Yuri LIMA AND Carlos MATHEUS

ABSTRACT. – This work constructs symbolic dynamics for non-uniformly hyperbolic surface maps with a set of discontinuities $\mathscr{D}$. We allow the derivative of points nearby $\mathscr{D}$ to be unbounded, of the order of a negative power of the distance to $\mathscr{D}$. Under natural geometrical assumptions on the underlying space M, we code a set of non-uniformly hyperbolic orbits that do not converge exponentially fast to $\mathscr{D}$. The results apply to non-uniformly hyperbolic planar billiards, e.g., Bunimovich billiards.

RÉSUMÉ. – Nous construisons une dynamique symbolique pour les applications non uniformément hyperboliques d'une surface ayant un ensemble de discontinuités $\mathscr{D}$. La dérivée de l'application peut ne pas être bornée, de l'ordre d'une puissance négative de la distance à $\mathscr{D}$. Sous certaines conditions géométriques naturelles sur l'espace des phases M, nous codifions un ensemble d'orbites non uniformément hyperboliques qui ne s'approchent pas exponentiellement vite de $\mathscr{D}$. Notre résultat s'applique aux billards planaires non uniformément hyperboliques tels que les billards de Bunimovich.

1. Introduction

Given a compact domain $T \subset \mathbb{R}^2$ with piecewise smooth boundary, consider the straight line motion of a particle inside T, with specular reflections in ∂T. Let $f : M \to M$ be the *billiard map*, where $M = \partial T \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ with the convention that $(r, \theta) \in M$ represents $r =$ collision position at ∂T and $\theta =$ angle of collision. The map f has a natural invariant Liouville measure $d\mu = \cos\theta dr d\theta$. Sinaĭ proved that dispersing billiards are uniformly hyperbolic systems with discontinuities [24], hence the Liouville measure is ergodic.

For a while uniform hyperbolicity was the only mechanism to generate chaotic billiards, until Bunimovich constructed examples of ergodic nowhere dispersing billiards [10, 11, 7]. These billiards, known as *Bunimovich billiards*, are non-uniformly hyperbolic: μ-almost every point has one positive Lyapunov exponent and one negative Lyapunov exponent, see [14, Chapter 8]. In this paper we construct symbolic models for non-uniformly hyperbolic billiard maps such as Bunimovich billiards. Assume that the billiard table T satisfies the conditions of [17, Part V], and let h be the Kolmogorov-Sinaĭ entropy of μ.

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THEOREM 1.1. – *If μ is ergodic and $h > 0$ then there exists a topological Markov shift (Σ, σ) and a Hölder continuous map $\pi : \Sigma \to M$ s.t.:*

(1) $\pi \circ \sigma = f \circ \pi$.
(2) *π is surjective and finite-to-one on a set of full μ-measure.*

Other examples of non-uniformly hyperbolic billiard maps are [25, 9]. See Section 1.3 for the definition of topological Markov shifts.

COROLLARY 1.2. – *Under the above assumptions, $\exists C > 0$ and $p \geq 1$ s.t. f has at least Ce^{hnp} periodic points of period np for all $n \geq 1$.*

Corollary 1.2 is consequence of Theorem 1.1 and the work of Gurevič [15, 16], as in [21, Thm. 1.1]. It is related to an estimate of Chernov [13]. The integer p is the period of (Σ, σ), hence $p = 1$ iff (Σ, σ) is topologically mixing. Since μ is mixing, we expect that the symbolic coding of Theorem 1.1 can be improved to give a topologically mixing (Σ, σ). Theorem 1.1 is consequence of the main result of this paper, Theorem 1.3, and of an argument of Katok and Strelcyn [17, Section I.3]. The statement of Theorem 1.3 is technical, so we first introduce some notation.

Let M be a smooth Riemannian surface with finite diameter, possibly with boundary. We assume that the diameter of M is smaller than one [(1)]. Let $\mathscr{D}^+, \mathscr{D}^-$ be closed subsets of M. Fix $f : M \backslash \mathscr{D}^+ \to M$ a diffeomorphism onto its image, s.t. f has an inverse $f^{-1} : M \backslash \mathscr{D}^- \to M$ that is a diffeomorphism onto its image.

Set of discontinuities $\mathscr{D}$. – The *set of discontinuities of f* is $\mathscr{D} := \mathscr{D}^+ \cup \mathscr{D}^-$.

If $x \notin \bigcup_{n \in \mathbb{Z}} f^n(\mathscr{D})$ then $f^n(x)$ is well-defined for all $n \in \mathbb{Z}$, and for every $y = f^n(x)$ there is a neighborhood $U \ni y$ s.t. $f \restriction_U, f^{-1} \restriction_U$ are diffeomorphisms onto their images. We require some regularity conditions on M, f. The first four assumptions are on the geometry of M. Given $x \in M \backslash \mathscr{D}$, let $\text{inj}(x)$ denote the injectivity radius of M at x, and let $\exp_x$ be the *exponential map* at x, wherever it can be defined. Given $r > 0$, let $B_x[r] \subset T_xM$ be the ball with center 0 and radius r. The Riemannian metric on M induces a Riemannian metric on TM, called the *Sasaki metric*, see e.g., [12, §2]. Denote the Sasaki metric by $d_{\text{Sas}}(\cdot, \cdot)$. Similarly, we denote the Sasaki metric on $TB_x[r]$ by the same notation, and the context will be clear in which space we are. For nearby small vectors, the Sasaki metric is almost a product metric in the following sense. Given a geodesic γ joining y to x, let $P_\gamma : T_yM \to T_xM$ be the parallel transport along γ. If $v \in T_xM$, $w \in T_yM$ then $d_{\text{Sas}}(v, w) \asymp d(x, y) + \|v - P_\gamma w\|$ as $d_{\text{Sas}}(v, w) \to 0$, see e.g., [12, Appendix A]. The rate of convergence depends on the curvature tensor of the metric on M. Here are the first two assumptions on M.

Regularity of $\exp_x$. – $\exists a > 1$ s.t. for all $x \in M \backslash \mathscr{D}$ there is $d(x, \mathscr{D})^a < \mathfrak{r}(x) < 1$ s.t. for $D_x := B(x, 2\mathfrak{r}(x))$ the following holds:

(A1) If $y \in D_x$ then $\text{inj}(y) \geq 2\mathfrak{r}(x)$, $\exp_y^{-1} : D_x \to T_yM$ is a diffeomorphism onto its image, and $\frac{1}{2}(d(x, y) + \|v - P_{y,x}w\|) \leq d_{\text{Sas}}(v, w) \leq 2(d(x, y) + \|v - P_{y,x}w\|)$ for all $y \in D_x$ and $v \in T_xM, w \in T_yM$ s.t. $\|v\|, \|w\| \leq 2\mathfrak{r}(x)$, where $P_{y,x} := P_\gamma$ for the radial geodesic γ joining y to x.

(1) Just multiply the metric by a sufficiently small constant.

(A2) If $y_1, y_2 \in D_x$ then $d(\exp_{y_1} v_1, \exp_{y_2} v_2) \leq 2d_{\text{Sas}}(v_1, v_2)$ for $\|v_1\|, \|v_2\| \leq 2\mathfrak{r}(x)$, and $d_{\text{Sas}}(\exp_{y_1}^{-1} z_1, \exp_{y_2}^{-1} z_2) \leq 2[d(y_1, y_2) + d(z_1, z_2)]$ for $z_1, z_2 \in D_x$ whenever the expression makes sense. In particular $\|d(\exp_x)_v\| \leq 2$ for $\|v\| \leq 2\mathfrak{r}(x)$, and $\|d(\exp_x^{-1})_y\| \leq 2$ for $y \in D_x$.

The next two assumptions are on the regularity of $d\exp_x$. For $x, x' \in M\backslash\mathscr{D}$, let $\mathscr{L}_{x,x'} := \{A : T_x M \to T_{x'} M : A \text{ is linear}\}$ and $\mathscr{L}_x := \mathscr{L}_{x,x}$. Then the parallel transport $P_{y,x}$ considered in (A1) is in $\mathscr{L}_{y,x}$. Given $y \in D_x, z \in D_{x'}$ and $A \in \mathscr{L}_{y,z}$, let $\widetilde{A} \in \mathscr{L}_{x,x'}$, $\widetilde{A} := P_{z,x'} \circ A \circ P_{x,y}$. By definition, $\widetilde{A}$ depends on x, x' but different base points define a map that differs from $\widetilde{A}$ by pre and post composition with isometries. In particular, $\|\widetilde{A}\|$ does not depend on the choice of x, x'. Similarly, if $A_i \in \mathscr{L}_{y_i,z_i}$ then $\|\widetilde{A_1} - \widetilde{A_2}\|$ does not depend on the choice of x, x'. Define the map $\tau = \tau_x : D_x \times D_x \to \mathscr{L}_x$ by $\tau(y, z) = \widetilde{d(\exp_y^{-1})_z}$, where we use the identification $T_v(T_y M) \cong T_y M$ for all $v \in T_y M$.

Regularity of $d\exp_x$. – The following holds:

(A3) If $y_1, y_2 \in D_x$ then $\|\widetilde{d(\exp_{y_1})_{v_1}} - \widetilde{d(\exp_{y_2})_{v_2}}\| \leq d(x, \mathscr{D})^{-a} d_{\text{Sas}}(v_1, v_2)$ for all $\|v_1\|, \|v_2\| \leq 2\mathfrak{r}(x)$, and $\|\tau(y_1, z_1) - \tau(y_2, z_2)\| \leq d(x, \mathscr{D})^{-a}[d(y_1, y_2) + d(z_1, z_2)]$ for all $z_1, z_2 \in D_x$.

(A4) If $y_1, y_2 \in D_x$ then the map $\tau(y_1, \cdot) - \tau(y_2, \cdot) : D_x \to \mathscr{L}_x$ has Lipschitz constant $\leq d(x, \mathscr{D})^{-a} d(y_1, y_2)$.

Conditions (A1)–(A2) guarantee that the exponential maps and their inverses are well-defined and have uniformly bounded Lipschitz constants in balls of radii $d(x, \mathscr{D})^a$. Condition (A3) controls the Lipschitz constants of the derivatives of these maps, and condition (A4) controls the Lipschitz constants of their second derivatives. Here are some cases when (A1)–(A4) are satisfied, in increasing order of generality:

– The curvature tensor R of M is globally bounded, e.g., when M is the phase space of a billiard map.
– $R, \nabla R, \nabla^2 R, \nabla^3 R$ grow at most polynomially fast with respect to the distance to $\mathscr{D}$, e.g., when M is a moduli space of curves equipped with the Weil-Petersson metric [12].

Now we discuss the assumptions on f.

Regularity of f. – There are constants $0 < \beta < 1 < b$ s.t. for all $x \in M\backslash\mathscr{D}$:

(A5) If $y \in D_x$ then $\|df_y^{\pm 1}\| \leq d(x, \mathscr{D})^{-b}$.

(A6) If $y_1, y_2 \in D_x$ and $f(y_1), f(y_2) \in D_{x'}$ then $\|\widetilde{df_{y_1}} - \widetilde{df_{y_2}}\| \leq \mathfrak{K} d(y_1, y_2)^\beta$, and if $y_1, y_2 \in D_x$ and $f^{-1}(y_1), f^{-1}(y_2) \in D_{x''}$ then $\|\widetilde{df_{y_1}^{-1}} - \widetilde{df_{y_2}^{-1}}\| \leq \mathfrak{K} d(y_1, y_2)^\beta$.

Although technical, conditions (A5)–(A6) hold in most cases of interest, e.g., if $\|df^{\pm 1}\|$, $\|d^2 f^{\pm 1}\|$ grow at most polynomially fast with respect to the distance to $\mathscr{D}$. We finally define the measures we code. Fix $\chi > 0$.

χ-hyperbolic measure. – An f-invariant probability measure on M is called *χ-hyperbolic* if μ-a.e. $x \in M$ has one Lyapunov exponent $> \chi$ and another $< -\chi$.