ISSN 0012-9593

quatrième série - tome 51 fascicule 4 juillet-août 2018

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Jacob VAN DEN BERG & Demeter KISS & Pierre NOLIN

Two-dimensional volume-frozen percolation: deconcentration and prevalence of mesoscopic clusters

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 540 €. Hors Europe : 595 € ($ 863). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 51, 2018, p. 1017 à 1084

TWO-DIMENSIONAL VOLUME-FROZEN PERCOLATION: DECONCENTRATION AND PREVALENCE OF MESOSCOPIC CLUSTERS

BY Jacob VAN DEN BERG, Demeter KISS AND Pierre NOLIN

Abstract. – Frozen percolation on the binary tree was introduced by Aldous [1], inspired by sol-gel transitions. We investigate a version of the model on the triangular lattice, where connected components stop growing ("freeze") as soon as they contain at least N vertices, where N is a (typically large) parameter.

For the process in certain finite domains, we show a "separation of scales" and use this to prove a deconcentration property. Then, for the full-plane process, we prove an accurate comparison to the process in suitable finite domains, and obtain that, with high probability (as $N \to \infty$), the origin belongs in the final configuration to a mesoscopic cluster, i.e., a cluster which contains many, but much fewer than N, vertices (and hence is non-frozen).

For this work we develop new interesting properties for near-critical percolation, including asymptotic formulas involving the percolation probability $\theta(p)$ and the characteristic length $L(p)$ as $p \searrow p_c$.

Résumé. – La percolation gelée sur l'arbre binaire a été introduite par Aldous [1], inspiré par les transitions sol-gel. Nous étudions une version de ce modèle sur le réseau triangulaire, pour laquelle les composantes connexes arrêtent de croître (« gèlent ») dès qu'elles contiennent au moins N sommets, où N est un paramètre (typiquement grand).

Pour le processus dans certains domaines finis, nous prouvons une « séparation d'échelles », et nous l'utilisons pour démontrer une propriété de déconcentration. Ensuite, pour le processus dans tout le plan, nous établissons une comparaison précise avec le processus dans des domaines finis adéquats, et nous obtenons qu'avec grande probabilité (lorsque $N \to \infty$), l'origine appartient, dans la configuration finale, à une composante connexe mésoscopique, c'est-à-dire, une composante qui contient un grand nombre de sommets, mais beaucoup moins que N (et qui est donc non-gelée).

Pour ce travail, nous développons de nouvelles propriétés intéressantes de la percolation presque-critique, en particulier des formules asymptotiques faisant intervenir la probabilité de percolation $\theta(p)$ et la longueur caractéristique $L(p)$ quand $p \searrow p_c$.

 doi:10.24033/asens.2371

1. Introduction

1.1. Frozen percolation

Frozen percolation is a growth process which was first introduced by Aldous [1] on the binary tree, motivated by sol-gel transitions [39]. Let us first describe it informally, on an infinite simple graph $G = (V, E)$, where the vertices may be interpreted as particles. We start with all edges closed (i.e., all particles are isolated), and we try to turn them open independently of each other: at some random time τ_e uniformly distributed between 0 and 1, the edge $e \in E$ becomes open if and only if it connects two finite open connected components (otherwise it just stays closed). In other words, a connected component grows until it becomes infinite (i.e., it gelates), at which time it just stops growing: we say that it freezes, which explains the name of the process. Apart from sol-gel transitions, one may think of other interpretations, e.g., population dynamics (group formation), and pattern formation in general. There are, somewhat surprisingly at first sight, also interesting connections with (and potential applications to) forest-fire models (at least in the two-dimensional setting, studied in this paper).

The existence of the frozen percolation process is not clear at all. In [1], Aldous studies the case when G is the infinite 3-regular tree, as well as the case of the planted binary tree (where all vertices have degree 3, except the root vertex which has degree 1): using the tree structure, which allows for explicit computations, he shows that the frozen percolation process does exist in these two cases (and that it exhibits a fascinating form of self-organized critical behavior). However, Benjamini and Schramm noticed soon after Aldous' paper that such a process does not exist on the square lattice $\mathbb{Z}^2$ (see also Remark (i) after Theorem 1 in [10]).

In order to circumvent this non-existence issue, a "truncated" process was studied in [8] by de Lima and two of the authors, where a connected component stops growing when it reaches a certain "size" N, where $N \geq 1$ is some parameter of the process. Formally, the original frozen percolation process corresponds to $N = \infty$, and one would like to understand what happens as $N \to \infty$, in view of the non-existence result.

When N is finite, "size" can have various meanings, and in [8], the size of a cluster is measured by its diameter. This diameter-frozen process was then further studied by the second author in [25], who established a precise description as $N \to \infty$, which, roughly speaking, can be summarized as follows. Let us fix some $K > 1$, and look at a square of side length KN (centered at 0): only finitely many frozen clusters appear (the probability that there are more than k such clusters decays exponentially in k, uniformly in N), and they all freeze in a near-critical window around the percolation threshold p_c. In particular, it is shown that the frozen clusters all look like near-critical percolation clusters, with total density converging to 0 as $N \to \infty$, and with high probability the origin does not belong to a frozen cluster: in the final configuration, a typical point is on a macroscopic non-frozen cluster, i.e., a cluster with diameter of order N, but smaller than N.

The truncated process on a binary tree is studied in [7], where it is shown that the final configuration is completely different: a typical point is either on a frozen cluster (i.e., with diameter $\geq N$), or on a microscopic one (with diameter $O(1)$), but one observes neither macroscopic non-frozen clusters, nor mesoscopic ones. Moreover, the way of measuring the size of a cluster does not really matter in this case: under mild hypotheses (see Theorem 2 in [7]), the process converges (in some weak sense) to Aldous' process as $N \to \infty$.

In [9] two of us returned to the case of a two-dimensional lattice, where this time the size of a cluster was measured by its volume, i.e., the number of vertices that it contains. In that paper, we studied large finite boxes with side length a function $m(N)$ of the parameter N. Using classical percolation techniques (e.g., Russo-Seymour-Welsh a-priori bounds, and Kesten's scaling relations [23]), we showed that there is a sequence of "length scales" $m_1(N), m_2(N), \dots$ at which an exceptional behavior occurs (see Section 1.2 for more information). This tells only a part of what is going on: for each fixed k, $m_k(N) \ll N$ (and actually, much smaller than N^{α} for some $\alpha < 1$), and [9] does not tell what happens for boxes with bigger length (and for the full-lattice process).

In the present paper, which uses and develops considerably more sophisticated results about the percolation phase transition, we explore this unknown "territory". Throughout the paper, we work with a site version of frozen percolation, on the triangular lattice $\mathbb{T}$ (we do this because site percolation on $\mathbb{T}$ is the planar percolation process for which the most precise results are known, as discussed below). This lattice has vertex set

$$V(\mathbb{T}) = \{x + ye^{\pi i/3} \in \mathbb{C} \, : \, x, y \in \mathbb{Z}\},$$

and edge set $E(\mathbb{T})$ obtained by connecting all pairs $u, v \in V(\mathbb{T})$ for which $\|u - v\|_2 = 1$. If $u, v \in V(\mathbb{T})$ are connected by an edge, that is, $(u, v) \in E(\mathbb{T})$, we say that u and v are neighbors, and we write $u \sim v$.

The independent site percolation process on $\mathbb{T}$ can be described as follows. We consider a family $(\tau_v)_{v \in V(\mathbb{T})}$ of i.i.d random variables, with uniform distribution on $[0, 1]$. For $p \in [0, 1]$, we say that a vertex v is p-black (resp. p-white) if $\tau_v \le p$ (resp. $\tau_v > p$). Then, the vertices are independently black or white, with respective probabilities p and $1 - p$. We denote by $\mathbb{P}_p$ the corresponding product measure. Vertices can be grouped into maximal connected components (clusters) of p-black sites and p-white sites, which defines a partition of $V(\mathbb{T})$. It is a celebrated result [22] that for all $p \le p_c := 1/2$, there is a.s. no infinite p-black cluster, while for $p > p_c$, there exists a.s. a unique infinite p-black cluster. We refer the reader to [21] for an introduction to percolation theory.

We now define the volume-frozen percolation process itself, based on the same collection $(\tau_v)_{v \in V(\mathbb{T})}$. For a subset $A \subseteq V(\mathbb{T})$, its volume is the number of vertices that it contains, denoted by $|A|$. Let $G = (V, E)$ be a subgraph of $\mathbb{T}$, and $N \ge 1$ be a fixed parameter. At time $t = 0$, we set all the vertices in V to be white, and as time t evolves from 0 to 1, each vertex $v \in V$ can become black at time $t = \tau_v$ only: it is allowed to do so if and only if all the black clusters touching v have a volume strictly smaller than N (otherwise, v stays white until the end, i.e time $t = 1$). That is, black clusters are allowed to grow until their volume is larger than or equal to N, when their growth is stopped: such a cluster is then said to be frozen. Observe that it might be the case that a cluster never reaches volume N, because it is trapped in a small region surrounded by frozen clusters. We say that a black vertex is frozen at a given time t if (at that time) it belongs to a frozen cluster (it is then frozen at all times $t' > t$). We use the notation $\mathbb{P}_N^{(G)}$ for the probability measure governing the process, and we omit the graph G used when it is clear from the context. Note that this process is well-defined: it can be seen as a finite range interacting particle system, thus general theory [31] provides