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AN OBSTRUCTION TO SMALL-TIME LOCAL NULL CONTROLLABILITY FOR A VISCOUS BURGERS' EQUATION

BY FRÉDÉRIC MARBACH

ABSTRACT. – In this work, we are interested in the small-time local null controllability for the viscous Burgers' equation $y_t - y_{xx} + yy_x = u(t)$ on a line segment, with null boundary conditions. The second-hand side is a scalar control playing a role similar to that of a pressure. In this setting, the classical Lie bracket necessary condition introduced by Sussmann fails to conclude. However, using a quadratic expansion of our system, we exhibit a second order obstruction to small-time local null controllability. This obstruction holds although the information propagation speed is infinite for the Burgers equation. Our obstruction involves the $H^{-5/4}$ norm of the control. The proof requires the careful derivation of an integral kernel operator and the estimation of residues by means of *weakly singular integral operator* estimates.

RÉSUMÉ. – Nous nous intéressons à la contrôlabilité locale en temps petit pour l'équation de Burgers visqueuse $y_t - y_{xx} + yy_x = u(t)$, posée sur un segment, avec des conditions de Dirichlet nulles au bord. Le terme source au second membre est un contrôle scalaire qui joue un rôle similaire à celui d'une pression. Dans ce contexte, la condition de crochet de Lie nécessaire classique introduite par Sussmann ne permet pas de conclure. Cependant, en utilisant un développement à l'ordre deux du système étudié, nous mettons en lumière une obstruction de nature quadratique à la contrôlabilité locale en temps petit. Cette obstruction tient alors même que la vitesse de propagation de l'information dans cette équation de Burgers est infinie. Elle fait intervenir la norme $H^{-5/4}$ du contrôle. La démonstration nécessite le calcul soigneux du noyau d'un opérateur intégral, ainsi que l'estimation d'opérateurs résiduels à l'aide de la théorie de régularité pour les *opérateurs intégraux faiblement singuliers*.

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1. Introduction

1.1. Description of the system and our main result

For $T > 0$ a small positive time, we consider the line segment $x \in [0, 1]$ and the following one-dimensional viscous Burgers' controlled system:

$$\begin{cases} y_t - y_{xx} + yy_x = u(t) & \text{in } (0,T) \times (0,1), \\ y(t,0) = 0 & \text{in } (0,T), \\ y(t,1) = 0 & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,1). \end{cases} \tag{1.1}$$

The scalar control $u \in L^2(0,T)$ plays a role somewhat similar to that of a pressure for multi-dimensional fluid systems. Unlike some other studies, our control term u depends only on time and not on the space variable. It is supported on the whole segment [0, 1]. For any initial data $y_0 \in H_0^1(0,1)$ and any fixed control $u \in L^2(0,T)$, it can be shown (see Lemma 7 below) that system (1.1) has a unique solution in the space $X_T = L^2((0,T); H^2(0,1)) \cap H^1((0,T); L^2(0,1))$. We are interested in the behavior of this system in the vicinity of the null equilibrium state.

Definition 1. – *We say that system* (1.1) *is* small-time locally null controllable *if, for any small time $T > 0$, for any small size of the control $\eta > 0$, there exists a region of size $\delta > 0$ such that:*

$$\forall y_0 \in H_0^1(0,1) \text{ s.t. } |y_0|_{H_0^1} \leq \delta, \exists u \in L^2(0,T) \text{ s.t. } |u|_2 \leq \eta \text{ and } y(T,\cdot) = 0, \tag{1.2}$$

where $y \in X_T$ is the solution to system (1.1) *with initial condition y_0 and control u.*

Theorem 1. – *System* (1.1) *is not* small-time locally null controllable. *Indeed, there exist $T, \eta > 0$ such that, for any $\delta > 0$, there exists $y_0 \in H_0^1(0,1)$ with $|y_0|_{H_0^1} \leq \delta$ such that, for any control $u \in L^2(0,T)$ with $|u|_2 \leq \eta$, the solution $y \in X_T$ to* (1.1) *satisfies $y(T,\cdot) \neq 0$.*

We will see in the sequel that our proof actually provides a stronger result. Indeed, we prove that, for small times and small controls, whatever the small initial data y_0, the state $y(t)$ drifts towards a fixed direction. Of course, this prevents small-time local null controllability as a direct consequence.

1.2. Motivation: small-time obstructions due to non-linearities

Most of the known obstructions to small-time local null controllability for control systems governed by partial differential equations are due to linear features.

1.2.1. *Linear obstructions.* – The most common cause of linear obstruction is the presence, in the evolution equation, of a finite speed of propagation (e.g., for wave or transport systems). As an example, let us consider the following transport control system:

$$
\begin{cases} y_t + My_x = 0 & \text{in } (0,T)\times(0,L), \\ y(t,0) = v_0(t) & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,L), \end{cases} \tag{1.3}
$$

where $T > 0$ is the total time, $M > 0$ the propagation speed and $L > 0$ the length of the domain. The control is the boundary data v_0. No condition is imposed at $x = 1$ since the characteristics flow out of the domain. For system (1.3), small-time local null controllability cannot hold. Indeed, even if the initial data y_0 is very small, the control is only propagated towards the right at speed M. Thus, if $T < L/M$, controllability does not hold. Of course, if $T \geq L/M$, the characteristics method allows to construct an explicit control to reach any final state y_1 at time T. We modify (1.3) with a small viscosity $\nu > 0$:

$$
\begin{cases} y_t - \nu y_{xx} + My_x = 0 & \text{in } (0,T)\times(0,L), \\ y(t,0) = v_0(t) & \text{in } (0,T), \\ y(t,1) = 0 & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,L). \end{cases} \tag{1.4}
$$

System (1.4) is small-time globally null controllable, for any $\nu > 0$ (but the cost of controllability explodes as $\nu \to 0$ if T is too small; see [26] for a precise study). Similarly, the underdetermined inviscid system:

$$
\begin{cases} y_t + yy_x = 0 & \text{in } (0,T)\times(0,L), \\ y(0,x) = y_0(x) & \text{in } (0,L) \end{cases} \tag{1.5}
$$

is not small-time locally null controllable (whatever choice is made as controlled boundary conditions at $x = 0$ and $x = 1$). Indeed, locally, we have $|y| \leq M$ with a small M. However, its viscous counterpart:

$$
\begin{cases} y_t - \nu y_{xx} + yy_x = 0 & \text{in } (0,T)\times(0,L), \\ y(t,0) = v_0(t) & \text{in } (0,T), \\ y(t,1) = 0 & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,L) \end{cases} \tag{1.6}
$$

is small-time locally null controllable for any $\nu > 0$ (see [36]).

Other linear features not linked to a finite propagation speed can also yield obstructions to small-time local null controllability; we refer to the recent works [5] for the Kolmogorov equation, [8] for Grushin-type equations, or [40] for the heat equation in a specific setting.

1.2.2. *Quadratic obstructions.* – Very few situations are known when the obstruction comes from the non-linearity of the partial differential equation governing the control system.

An example of such a system is the control of a quantum particle in a moving potential well (box). This is a bilinear controllability problem for the Schrödinger equation. For such system, it can be shown that large time controllability holds (see [4] if only the particle