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COMPLEX HYPERBOLIC VOLUME AND INTERSECTION OF BOUNDARY DIVISORS IN MODULI SPACES OF POINTED GENUS ZERO CURVES

BY VINCENT KOZIARZ AND DUC-MANH NGUYEN

ABSTRACT. – We show that the complex hyperbolic metrics defined by Deligne-Mostow and Thurston on $\mathcal{M}_{0,n}$ are singular Kähler-Einstein metrics when $\mathcal{M}_{0,n}$ is embedded in the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{0,n}$. As a consequence, we obtain a formula computing the volume of $\mathcal{M}_{0,n}$ with respect to these metrics using intersection of boundary divisors of $\overline{\mathcal{M}}_{0,n}$. In the case of rational weights, following an idea of Y. Kawamata, we show that these metrics actually represent the first Chern class of some line bundles on $\overline{\mathcal{M}}_{0,n}$, from which other formulas computing the same volumes are derived.

RÉSUMÉ. – Nous démontrons que les métriques hyperboliques complexes introduites par Deligne-Mostow et Thurston sur l'espace de modules de surfaces de Riemann de genre zéro avec n points marqués $\mathcal{M}_{0,n}$ sont des métriques Kähler-Einstein singulières sur la compactification de Deligne-Mumford-Knudsen $\overline{\mathcal{M}}_{0,n}$. Nous en déduisons des formules calculant le volume de $\mathcal{M}_{0,n}$ muni de ces métriques en fonction des nombres d'intersection des diviseurs de bord de $\overline{\mathcal{M}}_{0,n}$. De plus, lorsque les poids sont tous rationnels, en développant une idée de Y. Kawamata, nous montrons que ces métriques sont aussi des représentants de la première classe de Chern de certains fibrés en droites sur $\overline{\mathcal{M}}_{0,n}$, ce qui nous permet d'obtenir d'autres formules calculant les mêmes volumes.

1. Introduction

Let $n \geq 3$ and $\mathcal{M}_{0,n}$ be the moduli space of Riemann surfaces of genus 0 with n marked points. Let $\mu = (\mu_1, \dots, \mu_n)$ be real weights satisfying $0 < \mu_s < 1$ and $\sum \mu_s = 2$. Following ideas of E. Picard, P. Deligne and G. D. Mostow [5] constructed—for certain rational values of the μ_s's satisfying some integrality conditions—complex hyperbolic lattices which enable in particular to endow $\mathcal{M}_{0,n}$ with a complex hyperbolic metric Ω_μ. The volume of the corresponding orbifolds has been computed by several authors in some special cases when $n = 5$ (see e.g., [24, 20, 19, 13]).

A few years later, W. P. Thurston noticed [22] that for any n-uple of real weights satisfying the two simple conditions above, one can construct naturally a metric completion $\overline{\mathcal{M}}^{\mu}_{0,n}$

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of $(\mathcal{M}_{0,n}, \Omega_\mu)$, which can be endowed with a *cone manifold structure*. He observed in particular that $(\mathcal{M}_{0,n}, \Omega_\mu)$ always has finite volume (see Section 7.4 for our normalization of the metric and the volume element; we will use equally the notation Ω_μ for the metric and its associated Kähler form). In a more recent paper [18], C. T. McMullen proved a Gauss-Bonnet theorem for cone manifolds from which he derived a formula for the volume of $(\mathcal{M}_{0,n}, \Omega_\mu)$.

The main purpose in this paper is to investigate those complex hyperbolic metrics by using ideas coming from complex (algebraic) geometry with an approach in the spirit of Chapter 17 of [6]. We prove in particular that the extension by zero $\tilde{\Omega}_\mu$ of Ω_μ is a well defined closed positive current on the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{0,n}$ of $\mathcal{M}_{0,n}$, and that it is actually a *singular Kähler-Einstein metric* on $\overline{\mathcal{M}}_{0,n}$, associated with a boundary divisor that we make explicit. As a consequence, we show that the volume of $\mathcal{M}_{0,n}$ with respect to Ω_μ can be computed from the intersection numbers of boundary divisors in $\overline{\mathcal{M}}_{0,n}$.

In order to state more precisely our main results, we need a few basic facts about $\overline{\mathcal{M}}_{0,n}$ (see *e.g* [7, 16, 15, 1]). The moduli space $\mathcal{M}_{0,n}$ has complex dimension $N := n-3$ and its complement in the smooth variety $\overline{\mathcal{M}}_{0,n}$ is the union of finitely many divisors called *boundary divisors*, or *vital divisors*, each of which uniquely corresponds to a partition of $\{1,\dots,n\}$ into two subsets $I_0 \sqcup I_1$ such that $\min\{|I_0|,|I_1|\} \geq 2$, see [15] for instance. We will denote by $\mathcal{P}$ the set of partitions satisfying this condition. For each partition $\mathcal{S} := \{I_0, I_1\} \in \mathcal{P}$, we denote by $D_{\mathcal{S}}$ the corresponding divisor in $\overline{\mathcal{M}}_{0,n}$. Exchanging I_0 and I_1 if necessary, we will always assume that $\mu_{\mathcal{S}} := \sum_{s\in I_1} \mu_s \leq 1$ (in order to lighten the notation, we do not write explicitly the dependence of the coefficients $\mu_{\mathcal{S}}$ on μ).

For any $s \in \{1,\dots,n\}$, we also define the divisor class ψ_s on $\overline{\mathcal{M}}_{0,n}$ associated to the pullback of the relative cotangent bundle of the universal curve by the section corresponding to the s-th marked point.

Finally, if D is a divisor on $\overline{\mathcal{M}}_{0,n}$, D^N means as usual that we take the N-th self-intersection of D. Our main result concerns the cohomology class of $\tilde{\Omega}_\mu$.

THEOREM 1.1. – *Let $n \geq 4$ and $\mathcal{M}_{0,n}$ be the moduli space of Riemann surfaces of genus 0 with n marked points. Let $\mu = (\mu_1,\dots,\mu_n)$ be real weights satisfying $0 < \mu_s < 1$ and $\sum \mu_s = 2$. Let $D_\mu := \sum_{\mathcal{S}\in\mathcal{P}} \lambda_{\mathcal{S}} D_{\mathcal{S}}$ where*

$$\lambda_{\mathcal{S}} = (|I_1|-1)(\mu_{\mathcal{S}}-1)+1.$$

Let $\tilde{\Omega}_\mu$ be the current on $\overline{\mathcal{M}}_{0,n}$ defined by the extension by zero of Ω_μ. Then $\tilde{\Omega}_\mu$ is the Kähler form of a singular Kähler-Einstein metric for the pair $(\overline{\mathcal{M}}_{0,n}, D_\mu)$, hence $\tilde{\Omega}_\mu$ is a current which represents the same cohomology class as the $\mathbb{R}$-divisor $\frac{1}{N+1}(K_{\overline{\mathcal{M}}_{0,n}} + D_\mu)$, where $K_{\overline{\mathcal{M}}_{0,n}}$ is the canonical divisor of $\overline{\mathcal{M}}_{0,n}$. Moreover, the volume of $(\mathcal{M}_{0,n}, \Omega_\mu)$ satisfies

$$(1)\quad \mathrm{Vol}(\mathcal{M}_{0,n}, \Omega_\mu) := \int_{\mathcal{M}_{0,n}} \Omega_\mu^N = \frac{1}{(N+1)^N}\Big(K_{\overline{\mathcal{M}}_{0,n}} + D_\mu\Big)^N$$
$$= \frac{1}{(N+1)^N}\left(\sum_{\mathcal{S}}\Big(|I_1|-1\Big)\Big(\mu_{\mathcal{S}} - \frac{|I_1|}{N+2}\Big) D_{\mathcal{S}}\right)^N$$

$$= \frac{1}{2^N}\left(-\sum_{s=1}^{n}\mu_s\,\psi_s + \sum_{\mathcal{S}}\mu_{\mathcal{S}}\,D_{\mathcal{S}}\right)^N.$$

REMARK 1.2. – Formula (1) for the volume is not an immediate consequence of the fact that the divisor $K_{\overline{\mathcal{M}}_{0,n}} + D_\mu$ and the current $(N+1)\,\tilde{\Omega}_\mu$ represent the same cohomology class. In particular, it does not make sense in general to compute the power of a current and we need to control the behavior of Ω_μ^N near the boundary of $\overline{\mathcal{M}}_{0,n}$, see Proposition 7.1 for a more precise statement.

- There exists a completely explicit algorithm to calculate the intersection numbers of divisors of the type $D_{\mathcal{S}}$ (see [17] and Appendix A below), but doing the calculation by hand is rather involved. In Appendices B and C we compute $\mathrm{Vol}(\mathcal{M}_{0,5},\Omega_\mu)$ for all admissible weight vectors μ, and $\mathrm{Vol}(\mathcal{M}_{0,6},\Omega_\mu)$ for some examples of μ. As another application, the covolume of Deligne-Mostow lattices can be calculated with our formula and the help of a computer program by C. Faber which computes intersections of boundary divisors in $\overline{\mathcal{M}}_{0,n}$. In this way, we recover the results of [18, Table 1].
- In Corollary 7.5, we compare formula (1) with the one in [18, Th. 1.2]. The two approaches are different: in formula (1) we relate the volume to the top self-intersection of the "orbifold" first Chern class of $(\overline{\mathcal{M}}_{0,n}, D_\mu)$, while McMullen relates it to the *cone manifold Euler characteristic* of Thurston's completion $\overline{\mathcal{M}}_{0,n}^{\mu}$ of $\mathcal{M}_{0,n}$. Note that $\overline{\mathcal{M}}_{0,n}^{\mu}$ does not play any role in our proof. From the perspective of numerical computations, McMullen's formula is more practical since the Euler characteristic of $\overline{\mathcal{M}}_{0,n}^{\mu}$ can be calculated from a rather simple formula.

Our approach also sheds light on the relation between Thurston's completion $\overline{\mathcal{M}}_{0,n}^{\mu}$ of $\mathcal{M}_{0,n}$ and $\overline{\mathcal{M}}_{0,n}$. Recall that Thurston identified $\mathcal{M}_{0,n}$ with the space of flat surfaces homeomorphic to the sphere $\mathbb{S}^2$ having n conical singularities with cone angles given by $2\pi(1-\mu_s)$ up to rescaling. A stratum of $\overline{\mathcal{M}}_{0,n}^{\mu}$ consists of flat surfaces which are the limits when some clusters of singularities collapse into points. On the other hand, each stratum of $\overline{\mathcal{M}}_{0,n}$ is encoded by a tree whose vertices are labeled by the subsets in a partition of $\{1,\dots,n\}$. Every point in such a stratum represents a stable curve with several irreducible components. Among those components, there is a particular one that we call *μ-principal* whose definition depends on μ (see Section 5.1). To each stratum S of $\overline{\mathcal{M}}_{0,n}^{\mu}$, we have a corresponding stratum $\tilde{S}$ of $\overline{\mathcal{M}}_{0,n}$ such that, for any flat surface represented by a point in S, the underlying Riemann surface with punctures is isomorphic to the μ-principal component of the stable curves represented by some points in $\tilde{S}$. So in some sense, one can say that $\overline{\mathcal{M}}_{0,n}^{\mu}$ is obtained from $\overline{\mathcal{M}}_{0,n}$ by "contracting" every boundary stratum to its μ-principal factor.

In the case when there is no subset I of $\{1,\dots,n\}$ such that $\sum_{i\in I}\mu_i = 1$, $\overline{\mathcal{M}}_{0,n}^{\mu}$ is actually a compactification of $\mathcal{M}_{0,n}$. In the literature, one can find other compactifications of $\mathcal{M}_{0,n}$ which are different from the Deligne-Mumford-Knudsen one $\overline{\mathcal{M}}_{0,n}$ (see in particular the papers of B. Hasset [12] and D. I. Smyth [21]). These compactifications are contractions