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MORSE-SMALE SYSTEMS AND HORSESHOES FOR THREE DIMENSIONAL SINGULAR FLOWS

BY Shaobo GAN AND Dawei YANG

Abstract. – We prove that every three-dimensional vector field can be C^1 accumulated by Morse-Smale ones, or by ones with a transverse homoclinic intersection of some hyperbolic periodic orbit. In contrast to the case of diffeomorphisms [14], the main difficulty here is that we need to deal with singularities. We also make progress on another conjecture related to Palis in this paper.

Résumé. – Nous montrons que tout champ de vecteurs en dimension trois peut être accumulé en topologie C^1 ou bien par un champ Morse-Smale, ou bien par un champ possédant une intersection homocline transverse associée à une orbite périodique hyperbolique. Contrairement au cas des difféomorphismes [14], la principale difficulté ici consiste à traiter les singularités. Nous progressons également en direction d'une autre conjecture de Palis.

1. Introduction

1.1. The main result

One of the main subjects in differentiable dynamical systems is to describe the dynamics of "most" dynamical systems. These theories were established in the last century. See [2] for instance. An important progress is due to Peixoto [43]:

Theorem (Peixoto). – *Assume that M^2 is a closed surface. A C^1 vector field on M^2 is C^1 structurally stable iff it is Morse-Smale. Moreover, every vector field can be accumulated by structurally stable ones in the C^1 topology.*

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Smale was interested in the generalization of Peixoto's result and he asked whether Morse-Smale vector fields are dense in the space of vector fields. Levinson wrote Smale that one couldn't expect Morse-Smale systems to be dense generally. Essential ideas were contained in Levinson's paper which was inspired by work of Cartwright and Littlewood. See [2, 48]. Smale noticed the point and he constructed his famous horseshoe (for two dimensional diffeomorphisms or three-dimensional vector fields) [47] which shows that the dynamics may be very complicated and Morse-Smale systems would not be dense in the space of diffeomorphisms or vector fields. As in [2, Page 16]: "At that moment the world turned upside down…, and a new life began".

Actually, Poincaré found an important phenomenon in his famous work [44] on celestial mechanics, which was called a "doubly asymptotical solution". Nowadays mathematicians call it a *transverse homoclinic intersection*. Smale found that his horseshoe is closely related to transverse homoclinic intersections. Three classical results are known:

- Poincaré showed that transverse homoclinic intersections can survive under small perturbations. Moreover, if a system has one transverse homoclinic intersection, then it has infinitely many transverse homoclinic intersections [44].
- Birkhoff showed that if a plane system has one transverse homoclinic intersection, then it has infinitely many hyperbolic periodic orbits [5].
- Smale proved that the existence of a transverse homoclinic intersection is equivalent to the existence of a horseshoe [47].

In this paper, a *horseshoe* of a vector field is a hyperbolic set that is topologically equivalent to a suspension of full shift with two symbols. Hence there are two kinds of typical dynamical systems: Morse-Smale systems or systems with a horseshoe. Their dynamical behaviors are quite different:

- The dynamics of Morse-Smale systems is very simple: the chain recurrent set of a Morse-Smale system is a set which consists in finitely many hyperbolic periodic orbits or singularities. The topological entropy is robustly zero.
- The dynamics of a system with a horseshoe is very complicated: its chain recurrent set contains a non-trivial basic set with dense periodic orbits. The topological entropy is robustly positive.

Is there other typical dynamics beyond the above two ones? Palis formulated the idea for diffeomorphisms, and he conjectured that

CONJECTURE (Palis [40, 41, 42]). – *Every system can be approximated either by Morse-Smale systems or by systems exhibiting a horseshoe.*

In this paper, we manage to prove such kind of results for three dimensional vector fields.

THEOREM A (Main Theorem). – *Every three dimensional vector field can be C^1 approximated by Morse-Smale ones or by ones exhibiting a horseshoe. In other words, Morse-Smale vector fields and vector fields with a horseshoe form a C^1 open dense set in the space of three dimensional vector fields.*

Important progresses have been made for the conjecture of Palis for diffeomorphisms: in the C^1 topology, Pujals-Sambarino [45] proved it for two-dimensional diffeomorphisms (as a corollary of a stronger result); Bonatti-Gan-Wen [9] gave a proof for three-dimensional diffeomorphisms; and finally Crovisier [14] proved the conjecture for *any* dimensional diffeomorphisms.

Comparing with the diffeomorphism case, singularities of vector fields bring more difficulties. This prevents one to use some techniques of diffeomorphisms to singular vector fields, such as Crovisier's central model and Pujals-Sambarino's distortion arguments. By considering the sectional Poincaré maps of the flows, sometimes one can get some (not all) similar properties between d-dimensional vector fields and $(d-1)$-dimensional diffeomorphisms. But singular vector field displays different dynamics, e.g., the famous *Lorenz attractor* [29]. In the spirit of the Lorenz attractor, *geometric Lorenz attractors* ([1, 17, 18]) were constructed in a theoretical way. Roughly, a geometric Lorenz attractor is a robust attractor of a three-dimensional vector field, and it contains a hyperbolic singularity which is accumulated by hyperbolic periodic orbits in a robust way. The return map of a geometric Lorenz attractor has some discontinuous points. This fact gives extra difficulties when one wants to generalize Mañé's classical argument [30] to singular flows [(1)].

The Lorenz attractor is not hyperbolic because of the existence of a singularity. On the other hand, Mañé [30] showed that a robust attractor of a surface diffeomorphism is hyperbolic. This implies that the dynamics of 3-dimensional singular flows are different from 2-dimensional diffeomorphisms. Morales-Pacifico-Pujals [32, 31, 33] studied geometric Lorenz attractors in an abstract way. They found the right concept, i.e., *singular hyperbolicity*, to describe the weak hyperbolicity of the Lorenz attractor, and they proved that a robust transitive set of a three-dimensional vector field is singular hyperbolic. But the dynamics of singular hyperbolic sets are not as clear as hyperbolic sets. For instance, there is no shadowing lemma of Anosov-Bowen type.

The dynamics of general transitive sets with singularities for three-dimensional vector fields are even more unclear for us than singular hyperbolic sets, even if the transitive sets have some dominated splitting with respect to the linear Poincaré flow. These are the main difficulties that we encounter. For a non-trivial transitive set *without singularities* of a generic any dimensional vector field, one can adapt Crovisier's central model [14] to get a transverse homoclinic intersection of a hyperbolic periodic orbit.

Let us be more precise. Let M^d be a d-dimensional C^∞ compact Riemannian manifold without boundary. Denote by $\mathscr{X}^1(M^d)$ the space of C^1 vector fields on M^d. Given $X \in \mathscr{X}^1(M^d)$, denote by $\phi_t = \phi_t^X$ the C^1 flow generated by X and by $\Phi_t = \mathrm{d}\phi_t : TM^d \to TM^d$ the tangent flow on the tangent bundle TM^d. If $X(\sigma) = 0$, then σ is called a *singularity* of X. Other points are called *regular*. Let $\mathrm{Sing}(X)$ be the set of singularities of X. For a regular point p, if $\phi_t(p) = p$ for some $t > 0$, then p is called *periodic*. Let $\mathrm{Per}(X)$ be the set of periodic points of X. If $x \in \mathrm{Sing}(X) \cup \mathrm{Per}(X)$, then x is called a *critical point* of X and $\mathrm{Orb}(x)$ is called a *critical orbit* or *critical element* of X.

(1) Mañé's classical argument was generalized to the case of diffeomorphisms by [45] and to the case of non-singular flow by [4].