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ARGOS SEMINAR ON INTERSECTIONS OF MODULAR CORRESPONDENCES

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ARGOS SEMINAR
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Abstract. **—** This volume contains the written account of the Bonn seminar on arithmetic geometry 2003/2004. It gives a coherent exposition of the theory of intersections of modular correspondences. The focus of the seminar is the formula for the intersection number of arithmetic modular correspondences due to Gross and Keating. Other topics treated are Hurwitz's theorem on the intersection of modular correspondences over the field of complex numbers, and the relation of the arithmetic intersection numbers to Fourier coefficients of Siegel-Eisenstein series.

Also included is background material on one-dimensional formal groups and their endomorphisms, and on quadratic forms over the ring of p-adic integers.

Résumé **(Séminaire ARGOS sur les intersections de correspondances modulaires)**

Ce volume consiste des exposés faits dans le cadre du séminaire de géométrie arithmétique de Bonn en 2003/2004. Il donne une exposition systématique de la théorie des intersections de correspondances modulaires. Le but principal est la formule de Gross-Keating du nombre d'intersection de correspondances modulaires arithmétiques. Autres sujets traités sont le théorème de Hurwitz sur l'intersection de correspondances modulaires sur le corps des nombres complexes, et la relation des nombres d'intersection arithmétiques aux coefficients de Fourier des séries de Siegel-Eisenstein.

On a aussi inclus des rappels sur les groupes formels à un paramètre et leurs endomorphismes, et sur les formes quadratiques sur l'anneau des entiers p-adiques.

CONTENTS

1. FOREWORD

1. Motivation and main results

This book is based on the notes for the ΑΡΓΟΣ[(1)] seminar of the winter semester 2003/2004 at the University of Bonn. Its aim was to go through the paper *On the intersection of modular correspondences* by Gross and Keating [**GK**], and understand it thoroughly. This subject was chosen for three reasons. First of all, it was felt that the mathematics contained in this paper (and the papers on which Gross and Keating base their article) is extremely interesting, and has become even more important recently, due to the use that S. Kudla and others have made of these results. Secondly, thanks to the elementary methods employed in the proofs of the main theorems, the seminar provided a rapid access, even to a novice in the field, to a deep and sophisticated topic in arithmetic algebraic geometry. Thirdly, it was felt from the start that the literature on the subject was not easy to penetrate and that therefore the effort made by all speakers to master this material should not be lost, and that a written account of the seminar should be made available.

The origin of the topics treated in the seminar goes back to the 19$^{\text{th}}$ century. Let $j = j(\tau)$ be the elliptic modular function on the upper half plane. For $m \geq 1$ let $\varphi_m(j, j') \in \mathbb{Z}[j, j']$ be the classical *modular polynomial,* defined by

$$\varphi_m(j(\tau), j(\tau')) = \prod_{\substack{A \in \mathrm{M}_2(\mathbb{Z}) \\ \det(A)=m \\ \bmod SL_2(\mathbb{Z})}} (j(\tau) - j(A\tau')) \quad . \tag{1.1}$$

Kronecker and Hurwitz established a number of important properties of these polynomials, as for instance their factorization into irreducible factors. They also proved degree formulas like

$$\deg\ f_m = \sum_{dd'=m} \max(d, d') \quad , \tag{1.2}$$

where $f_m(j) = \varphi_m(j, j)$, for m not a square.

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(1) Acronym for **Ar**ithmetische **G**eometrie **O**ber**s**eminar.

From the point of view of the seminar, the interest in these results lies in the fact that they can be interpreted as giving intersection numbers on the complex surface $S_{\mathbb{C}} = \operatorname{Spec} \mathbb{C}[j, j']$. Let $T_{m,\mathbb{C}} \subseteq S_{\mathbb{C}}$ be the divisor defined by $\varphi_m = 0$. Then (1.2) can be interpreted as the intersection formula

$$(T_{m,\mathbb{C}} \cdot T_{1,\mathbb{C}}) = \sum_{dd'=m} \max(d, d') \quad , \tag{1.3}$$

if m is not a square. Here $(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}})$ is defined by

$$(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}}) = \dim_{\mathbb{C}} \mathbb{C}[j, j']/(\varphi_{m_1}, \varphi_{m_2}) \quad . \tag{1.4}$$

More generally, Hurwitz showed that the divisors $T_{m_1,\mathbb{C}}$ and $T_{m_2,\mathbb{C}}$ intersect properly on $S_{\mathbb{C}}$ if and only if $m_1 m_2$ is not a perfect square and gave an explicit expression for the intersection number $(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}})$. This in turn leads to the famous class number relations of Kronecker and Hurwitz.

Gross and Keating took up this classical subject by adding an arithmetic dimension to it. Instead of usual intersection numbers they consider arithmetic intersection numbers. Let $S = \operatorname{Spec} \mathbb{Z}[j, j']$, which we consider as an arithmetic threefold. Let T_m be the arithmetic divisor defined by $\varphi_m = 0$. The arithmetic intersection number is defined for any triple of positive integers m_1, m_2, m_3 by

$$(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) = \log \# \, \mathbb{Z}[j, j']/(\varphi_{m_1}, \varphi_{m_2}, \varphi_{m_3}) \quad . \tag{1.5}$$

Gross and Keating derive a criterion for when this number is finite and give in this case an explicit expression for it (see below). This result is the main focus of the present book. Let us state it from the point of view adopted in these notes. Let $\mathcal{M}$ be the moduli space of elliptic curves over $\operatorname{Spec} \mathbb{Z}$ (since we impose no level structure, $\mathcal{M}$ is not a scheme, but a Deligne-Mumford stack). Put $\mathcal{S} = \mathcal{M} \times_{\operatorname{Spec} \mathbb{Z}} \mathcal{M}$. For a positive integer m, let $\mathcal{T}_m$ be the moduli space of isogenies of elliptic curves $E \to E'$ of degree m. Then $\mathcal{T}_m$ maps by a finite unramified morphism to $\mathcal{M} \times \mathcal{M}$. From this point of view, the intersection number above should be interpreted as

$$\sum_p \log(p) \cdot \sum_{x \in \mathcal{X}(\overline{\mathbb{F}}_p)} \frac{1}{\# \operatorname{Aut}(x)} \lg \widehat{\mathcal{O}}_{\mathcal{X},x},$$

where we denote by $\mathcal{X}$ the triple fiber product of $\mathcal{T}_{m_1}$, $\mathcal{T}_{m_2}$, and $\mathcal{T}_{m_3}$ over $\mathcal{M} \times \mathcal{M}$. Here the weighting factor $\frac{1}{\#\operatorname{Aut}(x)}$ is due to the fact that $\mathcal{X}$ is a stack.

We now state the main results contained in this volume.

We denote by $S_{\mathbb{C}}$ resp. by $T_{m,\mathbb{C}}$ the base change of S resp. T_m to $\operatorname{Spec} \mathbb{C}$. The first result is Hurwitz's theorem.

Theorem 1.1. — *The cycles $T_{m_1,\mathbb{C}}$ and $T_{m_2,\mathbb{C}}$ intersect properly on $S_{\mathbb{C}}$ if and only if the integer $m = m_1 m_2$ is not a perfect square. In this case, the intersection $T_{m_1,\mathbb{C}} \times_{S_{\mathbb{C}}} T_{m_2,\mathbb{C}}$ lies over the locus in $S_{\mathbb{C}}$ corresponding to pairs (E, E') of elliptic curves with complex multiplication by orders in the imaginary-quadratic field $\mathbb{Q}(\sqrt{-m})$ of discriminant $\geq -4m$. The intersection number is equal to*

$$(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}}) = \sum_{\substack{t \in \mathbb{Z} \\ t^2 < 4m}} \sum_{d | gcd(m_1, m_2, t)} d \cdot H\left(\frac{4m - t^2}{d^2}\right) \quad . \quad \square$$

Here $H(n)$ denotes the Hurwitz class number (the number of $\mathrm{SL}_2(\mathbb{Z})$-equivalence classes of positive definite binary quadratic forms over $\mathbb{Z}$ with determinant n).

The second result is the theorem of Gross and Keating.

Theorem 1.2. — *The cycles T_{m_1}, T_{m_2} and T_{m_2} intersect properly on S if and only if there is no positive definite binary quadratic form over $\mathbb{Z}$ which represents the three integers m_1, m_2, m_3. In this case the intersection $T_{m_1} \times_S T_{m_2} \times_S T_{m_3}$ lies over the locus in S corresponding to pairs (E, E') of elliptic curves which are supersingular in some characteristic p with $p < 4m_1m_2m_3$. The arithmetic intersection number is equal to*

$$(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) = \sum_{p<4m_1m_2m_3} n(p) \log p \quad ,$$

where

$$n(p) = \frac{1}{2} \cdot \sum_{Q} \Bigg(\prod_{\substack{\ell \mid \Delta, \\ \ell \neq p}} \beta_\ell(Q) \Bigg) \cdot \alpha_p(Q) \quad . \qquad \square$$

Here the sum is the taken over all positive definite integral ternary quadratic forms Q with diagonal (m_1, m_2, m_3) which are isotropic over $\mathbb{Q}_\ell$ for all $\ell \neq p$. Furthermore $\Delta = \frac{1}{2} \det Q$ and $\beta_\ell(Q)$ is a normalized representation density of Q by the $\mathbb{Z}_\ell$-lattice $\mathrm{M}_2(\mathbb{Z}_\ell)$ with its norm form. Finally, $\alpha_p(Q)$ is the length of a certain local deformation space. Namely, one considers the universal deformation space of a triple of isogenies of formal groups of dimension 1 and height 2 over $\overline{\mathbb{F}}_p$. Here the passage from a global problem involving elliptic curves to a local problem on formal groups is provided by the Serre-Tate theorem. In [**GK**], Gross and Keating give completely explicit expressions for the factors $\beta_\ell(Q)$ and $\alpha_p(Q)$, comp. Chapters 5 and 13. They express these quantities in terms of new invariants of ternary quadratic forms over $\mathbb{Z}_p$ which are defined by them for this purpose (the Gross-Keating invariant in $(\mathbb{Z}_{\geq 0})^3$ and the Gross-Keating epsilon factor in $\{\pm 1\}$). This is especially striking in the cases when $\ell = 2$ resp. $p = 2$, in the other cases these invariants can be expressed in terms of classical quantities.

The invariant $\alpha_p(Q)$ is probably the most interesting ingredient in the formula above, and we now give a precise definition.

Let G be a formal group of dimension 1 and height 2 over $\bar{\mathbb{F}}_p$. Let $W = W(\bar{\mathbb{F}}_p)$ be the ring of Witt vectors. The universal deformation of the pairs (G, G) is then (Γ, Γ') over the formal scheme $\hat{\mathcal{S}} = \mathrm{Spf}\ W[[t, t']]$. If now $f_1, f_2, f_3 : G \to G$ are three endomorphisms $\neq 0$, we let $I_i \subset W[[t, t']]$ for $i = 1, 2, 3$ be the minimum ideal such that f_i lifts to a homomorphism $\tilde{f}_i : \Gamma \to \Gamma' \pmod{I_i}$. Then I_i defines a divisor $\hat{\mathcal{T}}_i$ on $\hat{\mathcal{S}}$. Consider

$$(\hat{\mathcal{T}}_1 \cdot \hat{\mathcal{T}}_2 \cdot \hat{\mathcal{T}}_3) = \mathrm{length}_W\ W[[t, t']]/(I_1 + I_2 + I_3) \quad . \tag{1.6}$$

On $\mathrm{End}(G)$ we have the usual quadratic form Nm with values in $\mathbb{Z}_p$ (the norm form, after identifying $\mathrm{End}(G)$ with the maximal order in the quaternion division algebra over $\mathbb{Q}_p$). It turns out that (1.6) only depends on the quadratic form $Q(f_1, f_2, f_3)$:

$(x, y, z) \mapsto \mathrm{Nm}(xf_1 + yf_2 + zf_3)$, and even only on its $\mathrm{GL}_3(\mathbb{Z}_p)$-equivalence class. We then set

$$\alpha_p(Q) = (\hat{\mathcal{T}}_1 \cdot \hat{\mathcal{T}}_2 \cdot \hat{\mathcal{T}}_3) \quad , \tag{1.7}$$

for any triple f_1, f_2, f_3 with $Q(f_1, f_2, f_3) = Q$. The formula for $\alpha_p(Q)$ in terms of the Gross-Keating invariant $(a_1, a_2, a_3) \in (\mathbb{Z}_{\geq 0})^3$ with $a_1 \leq a_2 \leq a_3$ is as follows. We note that for $p \neq 2$, in which case Q can be diagonalized, the integers a_1, a_2, a_3 are simply the p-adic valuations of the diagonal entries.

$$\begin{aligned}\alpha_p(Q) &= \sum_{i=0}^{a_1-1}(i+1)(a_1+a_2+a_3-3i)p^i + \sum_{i=a_1}^{(a_1+a_2-2)/2}(a_1+1)(2a_1+a_2+a_3-4i)p^i \\ &\quad + \frac{a_1+1}{2}(a_3-a_2+1)p^{(a_1+a_2)/2}, \text{ if } a_1 \equiv a_2 \pmod 2\end{aligned}$$

$$\begin{aligned}\alpha_p(Q) &= \sum_{i=0}^{a_1-1}(i+1)(a_1+a_2+a_3-3i)p^i + \sum_{i=a_1}^{(a_1+a_2-1)/2}(a_1+1)(2a_1+a_2+a_3-4i)p^i, \\ &\quad \text{if } a_1 \not\equiv a_2 \pmod 2\end{aligned}$$

The above results are the main focus of these notes. In the last chapter we reformulate Theorem 1.2 as a relation between the arithmetic intersection numbers and the Fourier coefficients of special values of derivatives of Siegel Eisenstein series, along the lines sketched in the introduction to [**GK**]. The idea that this can be done is attributed there to S. Kudla and D. Zagier; in the intervening years Kudla and others have gone a long way towards proving such relations in much greater generality [**Ku2, Ku3**]. Let

$$E(\tau, s) = \sum \det(\mathbf{c}\tau + \mathbf{d})^{-2} \cdot \frac{\det(y)^{\frac{s}{2}}}{|\det(\mathbf{c}\tau + \mathbf{d})|^s} \tag{1.8}$$

be the classical Siegel Eisenstein series of genus 3 and weight 2 for the full modular group. Here $\tau = x + iy \in \mathfrak{H}_3$ and $s \in \mathbb{C}$ is a complex parameter with large real part, and the sum is over representatives $\left(\gamma = \begin{smallmatrix} * & * \\ \mathbf{c} & \mathbf{d}\end{smallmatrix}\right)$ of the left cosets of the Siegel parabolic in $\mathrm{Sp}_3(\mathbb{Z})$. Then $E(\tau, s)$ has a meromorphic continuation to the entire s-plane and vanishes at $s = 0$. The derivative $E'(\tau, 0) = \frac{\partial E}{\partial s}(\tau, 0)$ is a non-holomorphic modular form of weight 2 for $\mathrm{Sp}_3(\mathbb{Z})$ and has a Fourier expansion

$$E'(\tau, 0) = \sum_{T \in \mathrm{Sym}_3(\mathbb{Z})^\vee} c'(T, y) \cdot q^T \quad , \tag{1.9}$$

where $q^T = \exp(2\pi i \ \mathrm{tr}(T\tau))$, for any half-integral matrix T. It turns out that for positive-definite T the coefficient $c'(T, y) \equiv c'(T)$ is independent of $y = \mathrm{Im}(\tau)$.

Theorem 1.3. — *Let m_1, m_2, m_3 be a triple of positive integers such that there is no positive definite binary quadratic form over $\mathbb{Z}$ which represents m_1, m_2 and m_3. There*