376

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LAGRANGIAN FLOER THEORY AND MIRROR SYMMETRY ON COMPACT TORIC MANIFOLDS

Kenji FUKAYA, Yong-Geun OH, Hiroshi OHTA & Kaoru ONO

Abstract. — In this paper we study Lagrangian Floer theory on toric manifolds from the point of view of mirror symmetry. We construct a natural isomorphism between the Frobenius manifold structures of the (big) quantum cohomology of the toric manifold and of Saito's theory of singularities of the potential function constructed in [Fukaya, *Tohoku Math. J.* **63** (2011)] via the Floer cohomology deformed by ambient cycles. Our proof of the isomorphism involves the open-closed Gromov-Witten theory of one-loop.

***Résumé* (La théorie de Floer lagrangienne ét la symétrie miroir sur les variétés toriques).** — Dans ce volume nous étudions la théorie de Floer lagrangienne sur les variétés toriques du point de vue de la symétrie miroir. Nous construisons un isomorphisme naturel entre les structures des variétés de Frobenius du grand anneau de cohomologie quantique de la variété torique et de la théorie des singularités de Saïto sur la fonction potentielle construite en [Fukaya, *Tohoku Math. J.* **63** (2011)] en utilisant la cohomologie de Floer déformée par les cycles ambiants. Notre démonstration de l'isomorphisme utilise les invariants de Gromov-Witten ouverts/fermés de la théorie d'une-boucle.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Introduction

The purpose of this paper is to prove a version of mirror symmetry between compact toric A-model and Landau-Ginzburg B-model.

Let X be a compact toric manifold and take a toric Kähler structure on it. In this paper X is not necessarily assumed to be Fano. In **[33]** we defined a potential function with bulk, $\mathfrak{PO}_{\mathfrak{b}}$, which is a family of functions of n variables $y_1, \dots, y_n$ parameterized by the cohomology class $\mathfrak{b} \in H(X;\Lambda_0)$. (More precisely, it is parameterized by T^n-invariant cycles. We explain this point later in Sections 1.3 and 2.5.) (Λ_0 is defined in Definition 1.2.1.)

$\mathfrak{PO}_{\mathfrak{b}}$ is an element of an appropriate completion of $\Lambda_0[y_1, y_1^{-1}, \dots, y_n, y_n^{-1}]$, the Laurent polynomial ring over (universal) Novikov ring Λ_0. We denote this completion by $\Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}}$. (See Definition 1.3.1 for its definition.) We put

$$\mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}}) = \frac{\Lambda\langle\langle y, y^{-1}\rangle\rangle_0^{\overset{\circ}{P}}}{\mathrm{Clos}_{d_{\overset{\circ}{P}}}\left(y_i \frac{\partial \mathfrak{PO}_{\mathfrak{b}}}{\partial y_i} : i = 1, \dots, n\right)}.$$

Differentiation of $\mathfrak{PO}_{\mathfrak{b}}$ with respect to the ambient cohomology class $\mathfrak{b}$ gives rise to a Λ_0 module homomorphism

$$\mathfrak{ks}_{\mathfrak{b}} : H(X;\Lambda_0) \to \mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}}). \tag{1.1.1}$$

We define a quantum ring structure $\cup^{\mathfrak{b}}$ on $H(X;\Lambda_0)$ by deforming the cup product by $\mathfrak{b}$ using Gromov-Witten theory. (See Definition 1.3.28.) The main result of this paper can be stated as follows:

Theorem 1.1.1. — *Equip* $H(X;\Lambda_0)$ *with the ring structure* $\cup^{\mathfrak{b}}$. *Then*

1. *The homomorphism* $\mathfrak{ks}_{\mathfrak{b}}$ *in* (1.1.1) *is a ring isomorphism.*
2. *If* $\mathfrak{PO}_{\mathfrak{b}}$ *is a Morse function, i.e., all its critical points are nondegenerate, the isomorphism* $\mathfrak{ks}_{\mathfrak{b}}$ *above intertwines the Poincaré duality pairing on* $H(X;\Lambda)$ *and the residue pairing on* $\mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}}) \otimes_{\Lambda_0} \Lambda$.

We refer readers to Section 1.3 for the definition of various notions appearing in Theorem 1.1.1. Especially the definition of residue pairing which we use in this paper is given in Definition 1.3.24 and Theorem 1.3.25.

We now explain how Theorem 1.1.1 can be regarded as a mirror symmetry between toric A-model and Landau-Ginzburg B-model:

(a) First, the surjectivity of (1.1.1) implies that the family $\mathfrak{PO}_{\mathfrak{b}}$ is a versal family in the sense of deformation theory of singularities. (See Theorem 2.8.1.)

(b) Therefore the tangent space $T_{\mathfrak{b}}H(X;\Lambda_0)$ carries a ring structure pulled-back from the Jacobian ring by (1.1.1). Theorem 1.1.1.1 implies that this pull-back coincides with the quantum ring structure $\cup^{\mathfrak{b}}$.

(c) Theorem 1.1.1.2 implies that the residue paring, which is defined when $\mathrm{Jac}(\mathfrak{PO}_{\mathfrak{b}})$ is a Morse function, can be extended to arbitrary class $\mathfrak{b} \in H(X;\Lambda_0)$. In fact, the Poincaré duality pairing of $H(X;\Lambda_0)$ is independent of $\mathfrak{b}$ and so obviously extended.

This (extended) residue pairing is nondegenerate and defines a 'Riemannian metric' on $H(X;\Lambda_0)$. (Here we put parenthesis since our metric is over the field of fractions Λ of Novikov ring and is not over $\mathbb{R}$.) As in the standard Riemannian geometry, it determines the Levi-Civita connection ∇ on $H(X;\Lambda_0)$. Theorem 1.1.1.2 then implies that this connection is flat: In fact, in the A-model side this connection is noting but the canonical affine connection on the affine space $H(X;\Lambda_0)$.

Let w_i be affine coordinates of $H(X;\Lambda_0)$ i.e., the coordinates satisfying $\nabla_{\partial/\partial w_i}\partial/\partial w_j = 0$. Then we have a function Φ on $H(X;\Lambda_0)$ that satisfies

$$\langle \mathbf{f}_i \cup^{\mathfrak{b}} \mathbf{f}_j, \mathbf{f}_k\rangle_{\mathrm{PD}_X} = \frac{\partial^3\Phi}{\partial w_i \partial w_j \partial w_k}. \tag{1.1.2}$$

Here $\{\mathbf{f}_i\}$ is the basis of $H(X;\Lambda_0)$ corresponding to the affine coordinates w_i and $\langle\cdot,\cdot\rangle_{\mathrm{PD}_X}$ is the Poincaré duality pairing.

In fact, Φ is constructed from Gromov-Witten invariants and is called Gromov-Witten potential. (See **[59]**, for example.) Using the isomorphism given in Theorem 1.1.1.1 we find that the third derivative of Φ also gives the structure constants of the Jacobian ring.

(d) In [**33**, Section 10], we defined an Euler vector field $\mathfrak{E}$ on $H(X;\Lambda_0)$. We also remark that the Jacobian ring carries a unit that is parallel with respect to the Levi-Civita connection: This is obvious in the A-model side and hence the same holds in the B-model side.

(e) The discussion above implies that the miniversal family $\mathfrak{PO}_{\mathfrak{b}}$ determines the structure of Frobenius manifold on $H(X;\Lambda_0)$. K. Saito **[63]** and M. Saito **[65]** defined a Frobenius manifold structure on the parameter space of miniversal