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SYMPLECTIC LIE GROUPS

Symplectic Reduction, Lagrangian extensions, and existence of Lagrangian normal subgroups

O. Baues, V. Cortés

***Abstract.* —** We develop the structure theory of symplectic Lie groups based on the study of their isotropic normal subgroups. The article consists of three main parts. In the first part we show that every symplectic Lie group admits a sequence of subsequent symplectic reductions to a unique irreducible symplectic Lie group. The second part concerns the symplectic geometry of cotangent symplectic Lie groups and the theory of Lagrangian extensions of flat Lie groups. In the third part of the article we analyze the existence problem for Lagrangian normal subgroups in nilpotent symplectic Lie groups.

***Résumé* (Groupes de Lie symplectiques. Réduction symplectique, extensions lagrangiennes et existence de sous-groupes lagrangiens distingués)**
On développe la théorie de la structure des groupes de Lie symplectiques basée sur l'étude de leurs sous-groupes distingués isotropes. L'article est constitué de trois parties principales. Dans la première partie, nous montrons que chaque groupe de Lie symplectique admet une séquence de réductions symplectiques successives à un unique groupe de Lie symplectique irréductible. La deuxième partie concerne la géométrie symplectique des groupes de Lie symplectiques cotangents et la théorie des extensions lagrangiennes des groupes de Lie plats. Dans la troisième partie de l'article, on fait une analyse du problème d'existence de sous-groupes distingués lagrangiens dans les groupes de Lie nilpotents symplectiques.

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CHAPTER 0

INTRODUCTION

A symplectic homogeneous manifold $(M = G/H, \omega)$ is a symplectic manifold (M, ω) which admits a transitive Lie group G of symplectic automorphisms. Symplectic homogeneous manifolds have been studied by many authors, see for instance [**43**] for a classification of compact symplectic homogeneous manifolds. Every symplectic homogeneous manifold of a given connected Lie group G can be described as follows: Let $\omega \in Z^2(\mathfrak{g})$ be a two-cocycle on the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$. Then $\mathfrak{h} = \ker \omega \subset \mathfrak{g}$ is a Lie subalgebra, which is contained in the stabilizer $\mathfrak{g}_\omega$ of ω under the coadjoint action of $\mathfrak{g}$ on forms. Let G_ω be the stabilizer of ω under the coadjoint action of G and assume that there exists a closed subgroup $H \subset G_\omega$ with Lie algebra $\mathfrak{h}$. Then $M = G/H$ is a smooth manifold and ω (considered as a two-form on $\mathfrak{g}/\mathfrak{h} \cong T_oM$, where $o = eH$ is the canonical base point) uniquely extends by the G-action to an invariant symplectic form ω on M. Notice that if $\mathfrak{g}$ is perfect, that is, if $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, then the Lie subgroup $H \subset G$ generated by $\mathfrak{h}$ is closed. In fact, in that case $H = (G_\omega)_0$. This happens, in particular, for semisimple Lie groups G, where all symplectic homogeneous manifolds $(M = G/H, \omega)$ are coadjoints orbits in $\mathfrak{g}^*$, which are endowed with their canonical symplectic form (which is unique up to scale). On the other hand, comparatively little is known about symplectic homogeneous manifolds of non-reductive groups G.

In this article we will concentrate on *symplectic Lie groups*, that is, on symplectic homogeneous manifolds with trivial stabilizer H. Thus a symplectic Lie group (G, ω) is a Lie group G endowed with a left-invariant symplectic form ω. The most important case is that of solvable Lie groups. In fact, it is known that unimodular symplectic Lie groups are *solvable*, see [**10**, Thm. 11]. For example, if G admits a *cocompact* discrete subgroup Γ then G belongs to this class. In this case, the coset-manifold $\Gamma\backslash G$, with symplectic structure induced by ω is a compact symplectic solvmanifold. Symplectic manifolds of this type have been of intense interest because they provide a rich source of non-Kähler compact symplectic manifolds [**38, 6, 5**]. Another important motivation to study symplectic homogeneous spaces of solvable Lie groups comes from their role in the theory of geometric quantization and the unitary representation theory of G as is apparent in the celebrated work of Kostant, Kirillov and Souriau (see, [**41**, Lecture 9] for a discussion).

A recurring theme in the study of symplectic Lie groups is their interaction with *flat* Lie groups. A flat Lie group is a Lie group endowed with a left-invariant torsion-free flat connection. For example, it is well known that every symplectic Lie group (G, ω) carries a torsion-free flat connection ∇^{ω} on G, which comes associated with the symplectic structure [**10**, Theorem 6]. Furthermore, flat Lie groups arise as quotients with respect to Lagrangian normal subgroups of symplectic Lie groups [**7**], and, more generally, in the context of reduction with respect to isotropic normal subgroups satisfying certain extra assumptions [**13**]. Another noteworthy appearance of flat Lie groups is as Lagrangian subgroups of symplectic Lie groups with the induced Weinstein connection.

These facts, which arise in the relatively rigid and restricted category of symplectic Lie groups, mirror more general constructions for arbitrary symplectic manifolds, in particular from the theory of Lagrangian and isotropic foliations [**14, 39, 40, 41**]. In fact, every *Lagrangian subgroup* L of a symplectic Lie group (G, ω) defines a left-invariant Lagrangian foliation on (G, ω). Therefore, a Lagrangian subgroup gives a left-invariant *polarization* in the sense of geometric quantization [**42**, Def. 4.5.1]. An important role is played by the method of *symplectic reduction* [**41**] adapted to the setting of symplectic Lie groups. Every normal isotropic subgroup of (G, ω) determines a coisotropic subgroup whose symplectic reduction is a symplectic Lie group. As in the general theory, foundational questions in the context of symplectic Lie groups are the *existence problem* for Lagrangian and isotropic subgroups as well as associated *classification* (that is, “normal form”) problems for symplectic Lie groups with Lagrangian subgroups.

There are several important contributions to the structure theory [**13, 12, 27, 34, 16**] and considerable classification work for symplectic Lie groups in low dimensions [**36, 23, 19, 26, 37, 17, 11, 35**]. However, the general picture concerning the above questions seems far from complete, with some of the main conjectures or research hypotheses remaining unverified in the literature. This concerns, in particular, the existence question for Lagrangian *normal* subgroups in completely solvable or nilpotent groups (first raised in [**7**]), the existence of Lagrangian subgroups in arbitrary symplectic Lie groups (with partial results given in [**13, 12, 18**]), and the theory of reduction with respect to general isotropic normal subgroups (where it remains to extend the approach pursued in [**13, 27**]), as well as various related or more specific questions (for example, in the context of nilpotent symplectic Lie groups as in [**20**] or [**19, 31, 15**]).

A natural class of symplectic Lie groups arises from Lie groups for which the coadjoint representation has an open orbit. The corresponding Lie algebras are called