

Junyi Xie

**THE DYNAMICAL MORDELL-LANG
CONJECTURE FOR POLYNOMIAL
ENDOMORPHISMS OF THE AFFINE
PLANE**

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THE DYNAMICAL MORDELL-LANG CONJECTURE
FOR POLYNOMIAL ENDOMORPHISMS OF THE
AFFINE PLANE

Junyi Xie

Abstract. — In this paper we prove the Dynamical Mordell-Lang Conjecture for polynomial endomorphisms of the affine plane over the algebraic numbers. More precisely, let f be an endomorphism of the affine plane over the algebraic numbers. Let x be a point in the affine plane and C be a curve. If the intersection of C and the orbits of x is infinite, then C is periodic.

Résumé (La conjecture de Mordell-Lang dynamique pour les applications polynomiales du plan affine)

Nous prouvons dans cet article la Conjecture Dynamique de Mordell-Lang pour les endomorphismes polynomiaux du plan affine sur les nombres algébriques. Plus précisément, soit f un endomorphisme du plan affine sur les nombres algébriques. Soient x un point dans le plan affine et C une courbe. Si l'intersection de C et les orbites de x est infinie, alors C est périodique.

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INTRODUCTION

1. THE DYNAMICAL MORDELL LANG CONJECTURE

This article is concerned with the so-called *dynamical Mordell-Lang conjecture* that was proposed by Ghioca and Tucker in [14].

DYNAMICAL MORDELL-LANG CONJECTURE (see [14]). — *Let X be a quasi-projective variety defined over \mathbb{C} , let $f : X \rightarrow X$ be an endomorphism, and V be any subvariety of X . For any point $p \in X(\mathbb{C})$ the set $\{n \in \mathbb{N} \mid f^n(p) \in V(\mathbb{C})\}$ is a union of at most finitely many arithmetic progressions⁽¹⁾.*

This conjecture is inspired by the Mordell-Lang conjecture on subvarieties of semiabelian varieties (now a theorem of Faltings [7] and Vojta [24]), which says that if V is a subvariety of a semiabelian variety G defined over \mathbb{C} and Γ is a finitely generated subgroup of $G(\mathbb{C})$, then $V(\mathbb{C}) \cap \Gamma$ is a union of at most finitely many translates of subgroups of Γ .

Observe that the dynamical Mordell-Lang conjecture implies the classical Mordell-Lang conjecture in the case $\Gamma \simeq (\mathbb{Z}, +)$.

It is also motivated by the Skolem-Mahler-Lech Theorem [21] on linear recurrence sequences. More precisely, suppose $\{A_n\}_{n \geq 0}$ is any recurrence sequence satisfying

$$A_{n+l} = F(A_n, \dots, A_{n+l-1})$$

for all $n \geq 0$, where $l \geq 1$ and $F(x_0, \dots, x_l) = \sum_{i=0}^{l-1} a_i x_i$ is a linear form on \mathbb{C}^l . The Skolem-Mahler-Lech Theorem asserts that the set $\{n \geq 0 \mid A_n = 0\}$ is a union of at most finitely many arithmetic progressions.

This statement is equivalent to the dynamical Mordell-Lang conjecture for the linear map $f : (x_0, \dots, x_{l-1}) \mapsto (x_1, \dots, x_{l-1}, F(x_0, \dots, x_l))$ and the hyperplane $V = \{x_0 = 0\}$.

1. An arithmetic progression is a set of the form $\{an + b \mid n \in \mathbb{N}\}$ with $a, b \in \mathbb{N}$. In particular, when $a = 0$, it contains only one point

2. THE MAIN RESULTS AND COMPARISON TO PREVIOUS RESULTS

Our goal is to prove this conjecture for *any* polynomial endomorphism of $\mathbb{A}_{\overline{\mathbb{Q}}}^2$.

THEOREM 1. — *Let $f : \mathbb{A}_{\overline{\mathbb{Q}}}^2 \rightarrow \mathbb{A}_{\overline{\mathbb{Q}}}^2$ be a polynomial endomorphism defined over $\overline{\mathbb{Q}}$. Let C be an irreducible curve in $\mathbb{A}_{\overline{\mathbb{Q}}}^2$ and p be a closed point in $\mathbb{A}_{\overline{\mathbb{Q}}}^2$. Then the set $\{n \in \mathbb{N} \mid f^n(p) \in C\}$ is a finite union of arithmetic progressions.*

Pick any polynomial $F(x, y) \in \overline{\mathbb{Q}}[x, y]$. By applying this result to the map

$$f : \mathbb{A}_{\overline{\mathbb{Q}}}^2 \longrightarrow \mathbb{A}_{\overline{\mathbb{Q}}}^2, \quad (x, y) \longmapsto (y, F(x, y))$$

and $C = \{x = 0\}$, we obtain the following corollary about recurrence sequences.

COROLLARY 2. — *Let $\{A_n\}_{n \geq 0}$ be a sequence of algebraic numbers satisfying*

$$A_{n+2} = F(A_n, A_{n+1})$$

for all $n \geq 0$, where $F(x, y) \in \overline{\mathbb{Q}}[x, y]$. Then the set $\{n \geq 0 \mid A_n = 0\}$ is a finite union of arithmetic progressions.

A direct induction on the dimension also yields the following

THEOREM 3. — *For any non-constant polynomials $F_1, \dots, F_m \in \overline{\mathbb{Q}}[T]$, let us consider on $\mathbb{A}_{\overline{\mathbb{Q}}}^m$ the endomorphism*

$$f := (F_1(x_1), \dots, F_m(x_m)).$$

For any irreducible curve $C \subset \mathbb{A}_{\overline{\mathbb{Q}}}^m$ defined over $\overline{\mathbb{Q}}$ and any point $p \in \mathbb{A}^m(\overline{\mathbb{Q}})$, the set

$$\{n \geq 0 \mid f^n(p) \in C\}$$

is a finite union of arithmetic progressions.

The dynamical Mordell-Lang conjecture has received quite a lot of attention in the recent years and our theorems are closely related to several known results.

Bell, Ghioca and Tucker [1] proved the Dynamical Mordell-Lang conjecture for étale maps of quasiprojective varieties of arbitrary dimension, thereby generalizing the Skolem-Mahler-Lech Theorem [21] on linear recurrence sequences. The core of their argument is to work in a p -adic field and to analyze the dynamics in a quasi-periodic region where they are able to construct suitable invariant curves. Afterwards, the author [27] proved the dynamical Mordell-Lang conjecture for birational endomorphisms of the affine plane. The techniques in [27] are of a very different flavour. Particularly, in [27], we obtained a new proof of the dynamical Mordell-Lang conjecture for polynomial automorphisms of \mathbb{A}^2 which are *not* conjugated to an automorphism of some projective surface. However, we relied on Bell, Ghioca and Tucker's result in some cases, especially in the case of affine automorphisms of \mathbb{A}^2 . In this paper, we develop the techniques used in [27] in a more general situation and use them more systematically.