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TILTING MODULES AND THE  $p$ -CANONICAL BASIS

Simon Riche & Geordie Williamson

**SOCIÉTÉ MATHÉMATIQUE DE FRANCE**

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# TILTING MODULES AND THE $p$ -CANONICAL BASIS

by Simon RICHE & Geordie WILLIAMSON

**Abstract.** — In this book we propose a new approach to tilting modules for reductive algebraic groups in positive characteristic. We conjecture that translation functors give an action of the (diagrammatic) Hecke category of the affine Weyl group on the principal block. Our conjecture implies character formulas for the simple and tilting modules in terms of the  $p$ -canonical basis, as well as a description of the principal block as the antispherical quotient of the Hecke category. We prove our conjecture for  $GL_n(\mathbb{k})$  using the theory of 2-Kac-Moody actions. Finally, we prove that the diagrammatic Hecke category of a general crystallographic Coxeter group may be described in terms of parity complexes on the flag variety of the corresponding Kac-Moody group.

**Résumé** — Dans cet ouvrage nous proposons une nouvelle approche à l'étude des modules basculants pour les groupes algébriques réductifs sur des corps de caractéristique positive. Nous conjecturons que les foncteurs de translation induisent une action de la catégorie de Hecke (diagrammatique) du groupe de Weyl affine sur le bloc principal. Cette conjecture implique des formules de caractères pour les modules simples et les modules basculants en termes de la base  $p$ -canonique, ainsi qu'une description du bloc principal comme le quotient anti-sphérique de la catégorie de Hecke. Nous démontrons notre conjecture pour le groupe  $GL_n(\mathbb{k})$  en utilisant la théorie des représentations des algèbres 2-Kac-Moody. Enfin, nous prouvons que la catégorie de Hecke diagrammatique d'un groupe de Coxeter cristallographique général peut être décrite en termes de faisceaux à parité sur la variété de drapeaux du groupe de Kac-Moody correspondant.



# CONTENTS

<b>1. Introduction</b> .....	1
1.1. Overview .....	1
1.2. The “categorical” conjecture .....	2
1.3. Tilting modules and the antispherical module .....	4
1.4. Tilting characters .....	6
1.5. The case of the group $GL_n(\mathbb{k})$ .....	8
1.6. The diagrammatic Hecke category and parity sheaves .....	9
1.7. Variants .....	10
1.8. Simple characters .....	11
1.9. Comparison with Lusztig’s conjecture .....	15
1.10. Acknowledgements .....	16
1.11. Organization of the book .....	17
<b>Part I. General conjecture</b> .....	19
<b>2. Tilting objects and sections of the <math>\nabla</math>-flag</b> .....	21
2.1. Highest weight categories .....	21
2.2. Canonical $\nabla$ -flags .....	22
2.3. Tilting objects and sections of the $\nabla$ -flag .....	23
<b>3. Regular and subregular blocks of reductive groups</b> .....	27
3.1. Definitions .....	27
3.2. Translation functors .....	29
3.3. Sections of the $\nabla$ -flag and translation to a wall .....	30
3.4. Sections of the $\nabla$ -flag and translation from a wall .....	33
3.5. Morphisms between “Bott-Samelson type” tilting modules .....	36
<b>4. Diagrammatic Hecke category and the antispherical module</b> .....	39
4.1. The affine Hecke algebra and the antispherical module .....	39
4.2. Diagrammatic Soergel bimodules .....	41
4.3. Two lemmas on $\mathcal{D}_{BS}$ .....	45
4.4. Categorified antispherical module .....	46
4.5. Morphisms in the categorified antispherical module .....	47
<b>5. Main conjecture and consequences</b> .....	53

5.1. Statement of the conjecture .....	53
5.2. Tilting modules and antispherical Soergel bimodules .....	55
5.3. Surjectivity .....	56
5.4. Dimensions of morphism spaces .....	59
5.5. Proof of Theorem 5.2.1 .....	60
5.6. Graded form of $\text{Rep}_0(G)$ .....	61
5.7. Integral form of $\text{Tilt}(\text{Rep}_0(G))$ .....	64
5.8. Integral form of $\text{Rep}_0(G)$ .....	66
<b>Part II. The case of <math>\text{GL}_n(\mathbb{k})</math> .....</b>	<b>73</b>
<b>6. Representations of <math>\text{GL}_n</math> in characteristic <math>p</math> as a 2-representation of <math>\widehat{\mathfrak{gl}}_p</math> .....</b>	<b>77</b>
6.1. The affine Lie algebra $\widehat{\mathfrak{gl}}_N$ .....	77
6.2. The natural representation of $\widehat{\mathfrak{gl}}_N$ .....	78
6.3. Realization of $\bigwedge^n \text{nat}_p$ as a Grothendieck group .....	79
6.4. $\text{Rep}(G)$ as a 2-representation .....	84
<b>7. Restriction of the representation to <math>\widehat{\mathfrak{gl}}_n</math> .....</b>	<b>93</b>
7.1. Combinatorics .....	93
7.2. Categorifying the combinatorics .....	94
7.3. First relations .....	95
7.4. Restriction of the 2-representation to $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ .....	100
<b>8. From categorical <math>\widehat{\mathfrak{gl}}_n</math>-actions to <math>\mathcal{D}_{\text{BS}}</math>-modules .....</b>	<b>103</b>
8.1. Strategy .....	103
8.2. Preliminary lemmas .....	105
8.3. Polynomials .....	107
8.4. One color relations .....	108
8.5. Cyclicity .....	109
8.6. Jones-Wenzl relations .....	110
8.7. Two color associativity .....	112
8.8. Zamolodchikov (three color) relations .....	117
<b>Part III. Relation to parity sheaves .....</b>	<b>125</b>
<b>9. Parity complexes on flag varieties .....</b>	<b>127</b>
9.1. Reminder on Kac-Moody groups and their flag varieties .....	127
9.2. Partial flag varieties .....	130
9.3. Derived categories of sheaves on $\mathcal{X}$ and $\mathcal{X}^s$ .....	132
9.4. Parity complexes on flag varieties .....	133
9.5. Sections of the !-flag .....	136
9.6. Sections of the !-flag and pushforward to $\mathcal{X}^s$ .....	137
9.7. Sections of the !-flag and pullback from $\mathcal{X}^s$ .....	140
9.8. Morphisms between “Bott-Samelson type” parity complexes .....	142



<b>10. Parity complexes and the Hecke category</b> .....	145
10.1. Diagrammatic category associated with $\mathcal{G}$ .....	145
10.2. More on Bott-Samelson parity complexes .....	146
10.3. Statement of the equivalences .....	147
10.4. Construction of the functor $\Delta_{\text{BS}}$ .....	148
10.5. Verification of the relations .....	152
10.6. Fully-faithfulness of $\Delta_{\text{BS}}$ .....	154
10.7. The case of the affine flag variety .....	158
<b>11. Whittaker sheaves and antispherical diagrammatic categories</b> .....	161
11.1. Definition of Whittaker sheaves .....	161
11.2. Whittaker parity complexes .....	163
11.3. Sections of the !-flag for Whittaker parity complexes .....	165
11.4. Surjectivity .....	167
11.5. Description of the antispherical diagrammatic category in terms of Whittaker sheaves .....	168
11.6. Application to the light leaves basis in the antispherical category ..	169
11.7. Iwahori-Whittaker sheaves on the affine flag variety .....	170
<b>List of notation</b> .....	173
<b>Bibliography</b> .....	179



# CHAPTER 1

## INTRODUCTION

### 1.1. Overview

In this book we give new conjectural character formulas for simple and indecomposable tilting modules for a connected reductive algebraic group in characteristic  $p$ ,<sup>(1)</sup> and we prove our conjectures in the case of the group  $\mathrm{GL}_n(\mathbb{k})$  when  $n \geq 3$ . These conjectures are formulated in terms of the  $p$ -canonical basis of the corresponding affine Hecke algebra. They should be regarded as evidence for the philosophy that Kazhdan-Lusztig polynomials should be replaced by  $p$ -Kazhdan-Lusztig polynomials in modular representation theory. From this point of view several conjectures (Lusztig's conjecture, James' conjecture, Andersen's conjecture) become the question of agreement between canonical (or Kazhdan-Lusztig) and  $p$ -canonical bases.

In the general setting we prove that the new character formulas follow from a very natural conjecture of a more categorical nature, which has remarkable structural consequences for the representation theory of reductive algebraic groups. It is a classical observation that wall-crossing functors provide an action of the affine Weyl group  $W$  on the Grothendieck group of the principal block; in this way, the principal block gives a categorification of the antispherical module for  $W$ . We conjecture that this action can be categorified: namely, that the action of wall-crossing functors on the principal block gives rise to an action of the diagrammatic Bott-Samelson Hecke category attached to  $W$  as in [31]. From this conjecture we deduce the following properties.

1. The principal block is equivalent (as a module category over the diagrammatic Bott-Samelson Hecke category) to a categorification of the antispherical module defined by diagrammatic generators and relations.
2. The principal block admits a grading. Moreover, this graded category arises via extension of scalars from a category defined over the integers. Thus the principal block of any reductive algebraic group admits a “graded integral form”.

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1. This book is written under the assumption that  $p$  is (strictly) larger than the Coxeter number, and so  $p$  cannot be too small. However there is a variant of our conjectures for any  $p$  involving singular variants of the Hecke category. In particular, it seems likely that  $p$ -Kazhdan-Lusztig polynomials give correct character formulas for tilting modules in *any* characteristic (see Conjecture 1.4.3). We hope to return to this subject in a future work.

3. The (graded) characters of the simple and tilting modules are determined by the  $p$ -canonical basis in the antispherical module for the Hecke algebra of  $W$ .

From (1) one may describe the principal block in terms of parity sheaves on the affine flag variety, which raises the possibility of calculating simple and tilting characters topologically. Point (2) gives a strong form of “independence of  $p$ ”. Finally, point (3) implies Lusztig’s character formula for large  $p$ .

We prove this “categorical” conjecture (hence in particular the character formulas) for the groups  $\mathrm{GL}_n(\mathbb{k})$  using the Khovanov-Lauda-Rouquier theory of 2-Kac-Moody algebra actions. We view this, together with the agreement with Lusztig’s conjecture for large  $p$  and character formulas of Soergel and Lusztig in the context of quantum groups (see §1.7 for details) as strong evidence for our conjecture.

## 1.2. The “categorical” conjecture

Let  $G$  be a connected reductive algebraic group over an algebraically closed field  $\mathbb{k}$  of characteristic  $p$  with simply connected derived subgroup. We assume that  $p > h$ , where  $h$  is the Coxeter number of  $G$ . Let  $T \subset B \subset G$  be a maximal torus and a Borel subgroup in  $G$ . Denote by  $\mathbf{X} := X^*(T)$  the lattice of characters of  $T$  and by  $\mathbf{X}^+ \subset \mathbf{X}$  the subset of dominant weights. Consider a regular block  $\mathrm{Rep}_0(G)$  of the category of finite-dimensional algebraic  $G$ -modules, corresponding to a weight  $\lambda_0 \in \mathbf{X}$  in the fundamental alcove, with its natural highest weight structure. If  $\Phi$  is the root system of  $(G, T)$ ,  $W_f = N_G(T)/T$  is the corresponding Weyl group, and  $W := W_f \ltimes \mathbb{Z}\Phi$  is the affine Weyl group, then the simple, standard, costandard and indecomposable tilting objects in the highest weight category  $\mathrm{Rep}_0(G)$  are all parametrized by  $W \bullet \lambda_0 \cap \mathbf{X}^+$ . (Here “ $\bullet$ ” denotes “ $p$ -dilated dot action of  $W$  of  $\mathbf{X}$ ,” see §3.1.) If  ${}^fW \subset W$  is the subset of elements  $w$  which are minimal in their coset  $W_f w$ , then there is a natural bijection

$${}^fW \xrightarrow{\sim} W \bullet \lambda_0 \cap \mathbf{X}^+$$

sending  $w$  to  $w \bullet \lambda_0$ . In this way we can parametrize the simple, standard, costandard and indecomposable tilting objects in  $\mathrm{Rep}_0(G)$  by  ${}^fW$ , and denote them by  $\mathbb{L}(x \bullet \lambda_0)$ ,  $\Delta(x \bullet \lambda_0)$ ,  $\nabla(x \bullet \lambda_0)$  and  $\mathbb{T}(x \bullet \lambda_0)$  respectively. In particular, on the level of Grothendieck groups we have

$$(1.2.1) \quad [\mathrm{Rep}_0(G)] = \bigoplus_{x \in {}^fW} \mathbb{Z}[\nabla(x \bullet \lambda_0)].$$

Let  $S \subset W$  denote the simple reflections. To any  $s \in S$  one can associate a “wall-crossing” functor  $\Xi_s$  by translating to and from an  $s$ -wall of the fundamental alcove. Consider the “antispherical” right  $\mathbb{Z}[W]$ -module  $\mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}[W_f]} \mathbb{Z}[W]$ , where  $\mathbb{Z}_\varepsilon$  denotes the sign module for the finite Weyl group  $W_f$  (viewed as a right  $\mathbb{Z}[W_f]$ -module). This module has a basis (as a  $\mathbb{Z}$ -module) consisting of the elements  $1 \otimes w$  with  $w \in {}^fW$ . Then we can reformulate (1.2.1) as an isomorphism

$$(1.2.2) \quad \phi: \mathbb{Z}_\varepsilon \otimes_{\mathbb{Z}[W_f]} \mathbb{Z}[W] \xrightarrow{\sim} [\mathrm{Rep}_0(G)]$$