

**Tye Lidman  
Ciprian Manolescu**

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**THE EQUIVALENCE OF TWO SEIBERG-WITTEN  
FLOER HOMOLOGIES**

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Diffusion

Maison de la SMF  
B.P. 67  
13274 Marseille Cedex 9  
France  
[christian.munusami@smf.emath.fr](mailto:christian.munusami@smf.emath.fr)

AMS  
P.O. Box 6248  
Providence RI 02940  
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[www.ams.org](http://www.ams.org)

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Secrétariat : Nathalie Christiaën

Astérisque

Société Mathématique de France

Institut Henri Poincaré, 11, rue Pierre et Marie Curie

75231 Paris Cedex 05, France

Tél : (33) 01 44 27 67 99

• Fax : (33) 01 40 46 90 96

[nathalie.christiaen@smf.emath.fr](mailto:nathalie.christiaen@smf.emath.fr)

• <http://smf.emath.fr/>

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T. LIDMAN

Department of Mathematics, North Carolina State University,  
Raleigh, NC 27607, USA.

*E-mail* : `tlid@math.ncsu.edu`

C. MANOLESCU

Department of Mathematics, UCLA, 520 Portola Plaza,  
Los Angeles, CA 90095, USA.

*E-mail* : `cm@math.ucla.edu`

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# THE EQUIVALENCE OF TWO SEIBERG-WITTEN FLOER HOMOLOGIES

Tye Lidman, Ciprian Manolescu

**Abstract.** — We show that monopole Floer homology (as defined by Kronheimer and Mrowka) is isomorphic to the  $S^1$ -equivariant homology of the Seiberg-Witten Floer spectrum constructed by the second author.

**Résumé (Équivalence des deux homologie de Seiberg-Witten Floer)**

Dans ce volume, nous montrons que l'homologie de Floer des monopoles (telle que définie par Kronheimer et Mrowka) est isomorphe à l'homologie  $S^1$ -équivariante du spectre de Seiberg-Witten Floer construit par le second auteur.



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## CHAPTER 1

### INTRODUCTION

#### 1.1. BACKGROUND

The Seiberg-Witten (monopole) equations [45], [46] are an important tool for understanding the topology of smooth four-dimensional manifolds. A signed count of the solutions of these equations on a closed four-manifold yields the Seiberg-Witten invariant [51]. On a four-manifold with boundary, instead of a numerical invariant one can define an element in a group associated to the boundary, called the Seiberg-Witten Floer homology. There are several different constructions of Seiberg-Witten Floer homology in the literature, [32], [28], [18], [13]. The goal of this monograph is to prove that, for rational homology spheres, the definitions given by Kronheimer-Mrowka in [18] and by the second author in [28] are equivalent.

The construction in [18] applies to an arbitrary three-manifold  $Y$ , equipped with a  $\text{Spin}^c$  structure  $s$ . Given this data, Kronheimer and Mrowka define an infinite dimensional analog of the Morse complex, with the underlying space being the blow-up of the configuration space of  $\text{Spin}^c$  connections and spinors (modulo gauge). The role of gradient flow lines is played by solutions to generic perturbations of the Seiberg-Witten equations on  $\mathbb{R} \times Y$ . Their invariant, monopole Floer homology, is the homology of the resulting complex. This complex (and hence also its homology) comes with a  $\mathbb{Z}[U]$ -module structure. The applications of monopole Floer homology include the surgery characterization of the unknot [19] and Taubes' proof of the Weinstein conjecture in three dimensions [48].

In fact, there are three different versions of monopole Floer homology defined in [18]; they are denoted  $\widetilde{HM}$ ,  $\widehat{HM}$ , and  $\overline{HM}$ . Yet another version,  $\widetilde{HM}$ , was constructed by Bloom in [3]: To define  $\widetilde{HM}$ , one considers the cone of the  $U$  map on the complex that defines  $\widehat{HM}$ , and then takes homology.

Compared with [18], the construction in [28] was originally done only for rational homology spheres; on the other hand, it yields something more than a homology group. By using finite dimensional approximation of the Seiberg-Witten equations, combined

with Conley index theory, one obtains an invariant in the form of an equivariant suspension spectrum. Specifically, given a rational homology sphere  $Y$  equipped with a  $\text{Spin}^c$  structure  $\mathfrak{s}$ , one can associate to it an  $S^1$ -equivariant spectrum  $\text{SWF}(Y, \mathfrak{s})$ . (See also [17], [16], [43] for extensions of this construction to the case  $b_1 > 0$ .)

The  $S^1$ -equivariant homology of  $\text{SWF}(Y, \mathfrak{s})$  can be viewed as a definition of Seiberg-Witten Floer homology. The advantage of having a Floer spectrum is that one can also apply other (equivariant) generalized homology functors to it. For example, by adding the conjugation symmetry, one can define a  $\text{Pin}(2)$ -equivariant Seiberg-Witten Floer homology; this was instrumental in the disproof of the triangulation conjecture by the second author [31]. For other applications of the Floer spectrum, see [29], [30], [27] and [26].

## 1.2. RESULTS

The bulk of this monograph is devoted to proving:

**THEOREM 1.1.** — *Let  $Y$  be a rational homology sphere with a  $\text{Spin}^c$  structure  $\mathfrak{s}$ . There is an isomorphism of absolutely-graded  $\mathbb{Z}[U]$ -modules:*

$$\widetilde{HM}_*(Y, \mathfrak{s}) \cong \widetilde{H}_*^{S^1}(\text{SWF}(Y, \mathfrak{s})),$$

where  $\widetilde{HM}$  is the “to” version of monopole Floer homology defined in [18], and  $\widetilde{H}_*^{S^1}$  denotes reduced equivariant (Borel) homology.<sup>1</sup>

From Theorem 1.1 we deduce that Bloom’s homology  $\widetilde{HM}$  can be identified with the ordinary (non-equivariant) homology of the Floer spectrum  $\text{SWF}$ :

**COROLLARY 1.2.** — *Let  $Y$  be a rational homology sphere equipped with a  $\text{Spin}^c$  structure  $\mathfrak{s}$ . Then,  $\widetilde{HM}_*(Y, \mathfrak{s}) \cong \widetilde{H}_*(\text{SWF}(Y, \mathfrak{s}))$  as absolutely graded abelian groups.*

From the absolute grading on monopole Floer homology one can extract a  $\mathbb{Q}$ -valued invariant, called the Frøyshov invariant; see [13] or [18, Section 39.1]. A similar numerical invariant, called  $\delta$ , was defined in [31, Section 3.7] using the Floer spectrum  $\text{SWF}$ . (The definition there was only given for  $\text{Spin}$  structures, but it extends to the  $\text{Spin}^c$  setting.) An immediate consequence of Theorem 1.1 is

**COROLLARY 1.3.** — *Let  $Y$  be a rational homology sphere equipped with a  $\text{Spin}^c$  structure  $\mathfrak{s}$ . Then,  $\delta(Y, \mathfrak{s}) = -h(Y, \mathfrak{s})$ , where  $h$  is the Frøyshov invariant as defined in [18, Section 39.1].*

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1. In this book, we grade Borel homology so that we simply have  $\widetilde{H}_*^{S^1}(X) = \widetilde{H}_*(X \wedge_{S^1} ES_+^1)$ . This differs from the grading conventions in [14] or [17] by one.