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THE EQUIVALENCE OF TWO SEIBERG-WITTEN FLOER HOMOLOGIES

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***Abstract.* —** We show that monopole Floer homology (as defined by Kronheimer and Mrowka) is isomorphic to the S^1-equivariant homology of the Seiberg-Witten Floer spectrum constructed by the second author.

***Résumé* (*Équivalence des deux homologie de Seiberg-Witten Floer*)**
Dans ce volume, nous montrons que l'homologie de Floer des monopoles (telle que définie par Kronheimer et Mrowka) est isomorphe à l'homologie S^1-équivariante du spectre de Seiberg-Witten Floer construit par le second auteur.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Background

The Seiberg-Witten (monopole) equations [**45**], [**46**] are an important tool for understanding the topology of smooth four-dimensional manifolds. A signed count of the solutions of these equations on a closed four-manifold yields the Seiberg-Witten invariant [**51**]. On a four-manifold with boundary, instead of a numerical invariant one can define an element in a group associated to the boundary, called the Seiberg-Witten Floer homology. There are several different constructions of Seiberg-Witten Floer homology in the literature, [**32**], [**28**], [**18**], [**13**]. The goal of this monograph is to prove that, for rational homology spheres, the definitions given by Kronheimer-Mrowka in [**18**] and by the second author in [**28**] are equivalent.

The construction in [**18**] applies to an arbitrary three-manifold Y, equipped with a Spin^c structure $\mathfrak{s}$. Given this data, Kronheimer and Mrowka define an infinite dimensional analog of the Morse complex, with the underlying space being the blow-up of the configuration space of Spin^c connections and spinors (modulo gauge). The role of gradient flow lines is played by solutions to generic perturbations of the Seiberg-Witten equations on $\mathbb{R} \times Y$. Their invariant, monopole Floer homology, is the homology of the resulting complex. This complex (and hence also its homology) comes with a $\mathbb{Z}[U]$-module structure. The applications of monopole Floer homology include the surgery characterization of the unknot [**19**] and Taubes' proof of the Weinstein conjecture in three dimensions [**48**].

In fact, there are three different versions of monopole Floer homology defined in [**18**]; they are denoted $\widecheck{HM}, \widehat{HM}$, and $\overline{HM}$. Yet another version, $\widetilde{HM}$, was constructed by Bloom in [**3**]: To define $\widetilde{HM}$, one considers the cone of the U map on the complex that defines $\widehat{HM}$, and then takes homology.

Compared with [**18**], the construction in [**28**] was originally done only for rational homology spheres; on the other hand, it yields something more than a homology group. By using finite dimensional approximation of the Seiberg-Witten equations, combined with Conley index theory, one obtains an invariant in the form of an equivariant suspension spectrum. Specifically, given a rational homology sphere Y equipped with a Spin^c structure $\mathfrak{s}$, one can associate to it an S^1-equivariant spectrum $\mathrm{SWF}(Y, \mathfrak{s})$. (See also [**17**], [**16**], [**43**] for extensions of this construction to the case $b_1 > 0$.)

The S^1-equivariant homology of $\mathrm{SWF}(Y, \mathfrak{s})$ can be viewed as a definition of Seiberg-Witten Floer homology. The advantage of having a Floer spectrum is that one can also apply other (equivariant) generalized homology functors to it. For example, by adding the conjugation symmetry, one can define a Pin(2)-equivariant Seiberg-Witten Floer homology; this was instrumental in the disproof of the triangulation conjecture by the second author [**31**]. For other applications of the Floer spectrum, see [**29**], [**30**], [**27**] and [**26**].

1.2. Results

The bulk of this monograph is devoted to proving:

Theorem 1.1. — *Let Y be a rational homology sphere with a* Spin^c *structure $\mathfrak{s}$. There is an isomorphism of absolutely-graded $\mathbb{Z}[U]$-modules:*

$$\widetilde{HM}_*(Y, \mathfrak{s}) \cong \widetilde{H}_*^{S^1}(\mathrm{SWF}(Y, \mathfrak{s})),$$

where $\widetilde{HM}$ is the "to" version of monopole Floer homology defined in [**18**]*, and $\widetilde{H}_*^{S^1}$ denotes reduced equivariant (Borel) homology.* [1]

From Theorem 1.1 we deduce that Bloom's homology $\widetilde{HM}$ can be identified with the ordinary (non-equivariant) homology of the Floer spectrum SWF:

Corollary 1.2. — *Let Y be a rational homology sphere equipped with a* Spin^c *structure $\mathfrak{s}$. Then, $\widetilde{HM}_*(Y, \mathfrak{s}) \cong \widetilde{H}_*(\mathrm{SWF}(Y, \mathfrak{s}))$ as absolutely graded abelian groups.*

From the absolute grading on monopole Floer homology one can extract a $\mathbb{Q}$-valued invariant, called the Frøyshov invariant; see [**13**] or [**18**, Section 39.1]. A similar numerical invariant, called δ, was defined in [**31**, Section 3.7] using the Floer spectrum SWF. (The definition there was only given for Spin structures, but it extends to the Spin^c setting.) An immediate consequence of Theorem 1.1 is

Corollary 1.3. — *Let Y be a rational homology sphere equipped with a Spin^c structure $\mathfrak{s}$. Then, $\delta(Y, \mathfrak{s}) = -h(Y, \mathfrak{s})$, where h is the Frøyshov invariant as defined in* [**18**, Section 39.1].

1. In this book, we grade Borel homology so that we simply have $\widetilde{H}_*^{S^1}(X) = \widetilde{H}_*(X \wedge_{S^1} ES^1_+)$. This differs from the grading conventions in [**14**] or [**17**] by one.