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GEOMETRIC QUANTIZATION AND ASYMPTOTICS OF PAIRINGS IN TQFT

BY RENAUD DETCHERRY

ABSTRACT. – This paper presents an explicit mapping between the SU(2)-Reshetikhin-Turaev TQFT vector spaces $V_r(\Sigma)$ of surfaces and spaces of holomorphic sections of complex line bundles on some Kähler manifold, following the approach of geometric quantization. We explain how curve operators in TQFT correspond to Toeplitz operators with symbols some trace functions. As an application, we show that eigenvectors of these operators are concentrated near the level sets of these trace functions, and obtain asymptotic estimates of pairings of such eigenvectors. This yields under some genericity assumptions an asymptotic for the matrix coefficients of quantum representations.

RÉSUMÉ. – Dans ce papier, nous construisons un isomorphisme explicite entre les espaces vectoriels $V_r(\Sigma)$ des TQTC de Reshetikhin-Turaev de groupe de gauge SU(2) et des espaces de sections holomorphes de fibrés en droites complexes sur une certaine variété kählerienne, suivant l'approche de la quantification géométrique. Les opérateurs courbes deviennent ainsi des opérateurs de Toeplitz de symboles principaux correspondant aux fonctions traces sur l'espace des modules. Nous en déduisons que les vecteurs propres de ces opérateurs se concentrent sur les lignes de niveaux de ces fonctions traces, et obtenons une formule asymptotique pour les produits scalaires de ces vecteurs propres. Ceci permet d'obtenir une asymptotique pour les coefficients de matrice des représentations quantiques satisfaisant une hypothèse de généricité.

1. Introduction

The study of topological quantum field theories (or TQFT) was developed after Witten used the Jones polynomial to heuristically define a collection of invariants of 3-manifolds, cobordisms, surfaces and mapping class on surfaces, satisfying some axioms, including some compatibility with gluings or disjoint union, and gave the expected asymptotic expansion of these 3-manifold invariants, in what is known as the Witten conjecture.

Later, these TQFT were constructed more rigorously by Reshetikhin and Turaev in [28], and later by Blanchet, Masbaum, Habegger and Vogel in [9], in the case where the gauge group is $G = \mathrm{SU}_2$, using skein calculus. This second approach, more combinatorial, is the one we use in the following paper.

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To each compact oriented surface Σ is associated by the TQFT a sequence of Hermitian vector spaces $V_r(\Sigma)$, parametrized by an integer r called level, to each pair of pants decomposition a basis $(\varphi_\alpha)_{\alpha\in I_r}$ of $V_r(\Sigma)$, and to each simple closed curve on Σ a curve operator $T_r^\gamma \in \mathrm{End}(V_r(\Sigma))$. The goal of this paper is to compute the asymptotic behavior of pairings $\langle\varphi_\alpha, \psi_\beta\rangle$ of basis vectors corresponding to two pants decomposition when the level goes to infinity.

A helpful tool to study asymptotics of quantum invariants is the theory of geometric quantization. Given a Kähler manifold (M,ω,J) with $\dim(M) = 2n$, we define a prequantization bundle as a complex line bundle with an Hermitian form h that has Chern curvature $\frac{1}{i}\omega$ and a half-form bundle, that is a square root of the bundle of complex n-forms. For L a prequantization bundle and δ a half-form bundle, we have a sequence of (finite dimensional when M is compact) vector spaces $H^0(M, L^r \otimes \delta)$: the spaces of holomorphic sections of $L^r \otimes \delta$.

A natural candidate to present the vector space $V_r(\Sigma)$ and curve operators T_r^γ as arising from the geometric quantization of some Kähler manifold and function is the moduli space $\mathcal{M}(\Sigma) = \mathrm{Hom}(\pi_1\Sigma, \mathrm{SU}_2)/\,\mathrm{SU}_2$ of representations of the fundamental group of Σ in SU_2 modulo conjugation. This space has a natural symplectic form on it, defined by Atiyah and Bott in [7].

Also, in the setting of geometric quantization, to each smooth integrable function f on $\mathcal{M}(\Sigma)$ is associated a sequence of endomorphisms of $H^0(M, L^r \otimes \delta)$ called a Toeplitz operator of symbol f. Curve operators T_r^γ will be represented as Toeplitz operators with principal symbol the trace functions $f_\gamma(\rho) = -\mathrm{Tr}(\rho(\gamma))$ which are continuous functions on $\mathcal{M}(\Sigma)$.

Then, results of microlocal analysis state that the joint eigenvectors of such Toeplitz operators concentrate on the level sets of their principal symbol. As the TQFT basis $(\varphi_\alpha)_{\alpha\in I_r}$ associated to a pair of pants decomposition $\mathcal{C}$ of Σ is a basis of common eigenvectors of curve operators, we get asymptotic estimates of these. In the geometric model, sequences of vectors φ_{α_r} with $\frac{\alpha_r}{r} \xrightarrow[r\to+\infty]{} x$ should carry most of their mass on a neighborhood of the set

$$\Lambda^{\mathcal{C}}_{-2\cos(\pi x)} = \{\rho \,/\, \mathrm{Tr}(\rho(C_i)) = -2\cos(\pi x_i)\}.$$

The final goal of this paper is to compute the asymptotic expansion of pairings $\langle\varphi_\alpha, \psi_\beta\rangle$ of basis vectors of $V_r(\Sigma)$ corresponding to two pants decompositions $\mathcal{C}$ and $\mathcal{D}$ of Σ. There are two ways of thinking of these pairings: if one decomposition is the image of the other by an element of the mapping class group Γ_g of Σ, what we compute is a limit of matrix coefficients of the quantum representations of Γ_g on $V_r(\Sigma)$.

Alternatively, the pairings can be viewed as special Reshethikhin-Turaev invariants: choose two handlebodies, one corresponding to each pants decomposition, insert a trivalent colored graph in each with colorings α and β, and glue them together to obtain a 3-manifold with a pair of trivalent-colored graphs inside, which represents some linear combination of links. The pairing $\langle\varphi_\alpha, \psi_\beta\rangle$ is just the Reshetikhin-Turaev invariant of this manifold with links.

The vectors φ_{α_r} and ψ_{β_r} concentrate on $\Lambda^{\mathcal{C}}_{E_{\alpha_r}}$ and $\Lambda^{\mathcal{D}}_{E_{\beta_r}}$ where $E^i_{\alpha_r} = -2\cos(\pi\frac{\alpha^i_r}{r})$. Under some condition of genericity, the Lagrangian $\Lambda^{\mathcal{C}}_x$ and $\Lambda^{\mathcal{D}}_y$ have a transverse intersection, consisting only of a finite number of points, and we show that the pairing $\langle\varphi_{\alpha_r},\psi_{\beta_r}\rangle$ has an asymptotic expansion consisting of a sum of contributions of these points as follows:

$$\langle\varphi_{\alpha_r},\psi_{\beta_r}\rangle = u_r(\frac{r}{2\pi})^{-\frac{n}{2}}\frac{1}{\sqrt{\mathrm{Vol}(\Lambda^{\mathcal{C}}_{E_{\alpha_r}})\mathrm{Vol}(\Lambda^{\mathcal{D}}_{E'_{\beta_r}})}}\sum_{z\in\Lambda^{\mathcal{C}}_{E_{\alpha_r}}\cap\Lambda^{\mathcal{D}}_{E'_{\beta_r}}}\frac{e^{ir\eta(z_0,z)}}{|\det(\{\mu_i,\mu'_j\})|^{\frac{1}{2}}}i^{m(z_0,z)} + O(r^{-\frac{n}{2}-1}),$$

where we set $\dim(\mathcal{M}(\Sigma)) = 2n$ (that is $n = 3g-3$), u_r is a sequence of complex numbers of moduli 1, the functions $\mu_i = -\mathrm{Tr}(\rho(C_i))$ and $\mu'_j = -\mathrm{Tr}(\rho(D_j))$ are the principal symbols of $T_r^{C_i}$ and $T_r^{D_j}$, $\{\cdot,\cdot\}$ is the Poisson bracket in $\mathcal{M}(\Sigma)$, $Vol(\Lambda^{\mathcal{C}}_{E_{\alpha_r}})$ (resp. $Vol(\Lambda^{\mathcal{D}}_{E_{\beta_r}})$) is the volume for the volume form $\beta = d\theta_1\wedge\cdots\wedge d\theta_n$ (resp. $\beta' = d\theta'_1\wedge\cdots d\theta'_n$) where $d\theta_i$ (resp. $d\theta'_i$) are basis of $T^*\Lambda^{\mathcal{C}}_{E_{\alpha_r}}$ (resp. $T^*\Lambda^{\mathcal{D}}_{E_{\beta_r}}$) dual to the Hamiltonian vector fields X_i of μ_i (resp. X'_i of μ'_i).

Moreover, z_0 is some specific point in the finite intersection, and for $z\in\Lambda^{\mathcal{C}}_{E_{\alpha_r}}\cap\Lambda^{\mathcal{C}}_{E_{\alpha_r}}$, if we choose $\gamma_{z_0,z}$ a loop consisting of a path from z_0 to z in $\Lambda^{\mathcal{C}}_{E_{\alpha_r}}$ and a path from z to z_0 in $\Lambda^{\mathcal{D}}_{E_{\beta_r}}$, then $\eta(z_0,z)$ is the holonomy of the prequantization bundle L along $\gamma_{z_0,z}$. The line bundle L having curvature $\frac{1}{i}\omega$, the quantity $\eta(z_0,z)$ can be defined alternatively as follows: take an oriented disk $D(z_0,z)$ in $\mathcal{M}(\Sigma)$ whose boundary is the loop $\gamma_{z_0,z}$. Then $\eta(z_0,z)$ is its symplectic area: $\eta(z_0,z) = \int_{D(z_0,z)}\omega$

Finally, $m(z_0,z)\in\mathbb{Z}$ and is some kind of Maslov index, the computation of which is explained in Section 5. A path from z_0 to z along $\Lambda^{\mathcal{C}}_{E_{\alpha_r}}$ induces a path in the oriented Lagrangian Grassmanian $LG^+(\mathcal{M}(\Sigma))$, similarly for the path from z to z_0 in $\Lambda^{\mathcal{C}}_{E_{\beta_r}}$. We connect these to get a loop in the oriented Lagrangian Grassmanian by turning "positively" in the oriented Lagrangian Grassmanian of $T_z\mathcal{M}(\Sigma)$ and $T_{z_0}\mathcal{M}(\Sigma)$. The index $m(z_0,z)$ ought to correspond the homotopy class of this loop in $\pi_1 LG^+(\mathcal{M}(\Sigma)) = \mathbb{Z}$.

Notice that the definition of $\eta(z_0,z)$ depends only on the homotopy class of the loop $\gamma_{z_0,z}$ as the line bundle L is a flat bundle on the Lagrangian $\Lambda^{\mathcal{C}}_{E_{\alpha_r}}$ and $\Lambda^{\mathcal{D}}_{E_{\beta_r}}$. Actually, the quantity $r\eta(z_0,z)+\frac{\pi}{2}m(z_0,z)$ will be also independent of this homotopy class modulo $2\pi\mathbb{Z}$, as a result of Bohr-Sommerfeld conditions.

Finally, we note that the first term of the asymptotic expansion of this pairing is defined uniquely from the positions of the Lagrangian $\Lambda^{\mathcal{C}}_{E_{\alpha_r}}$ and $\Lambda^{\mathcal{D}}_{E_{\beta_r}}$ and the symplectic structure of $\mathcal{M}(\Sigma)$, so it does not depend on the complex structure we introduced in the process of geometric quantization.

The idea of linking curve operators in TQFT to Toeplitz operators originates in the work of Andersen [1]. Andersen works in the geometrical viewpoint of TQFT, representing TQFT vector spaces $V_r(\Sigma)$ as spaces $V_r^\sigma(\Sigma)$ of holomorphic sections on the moduli space depending on the choice of a complex structure σ on Σ, and introduces in [1] some Toeplitz operators with trace functions as principal symbols, and shows that they approximate curve operators at first order. This approach proved rapidly fruitful: Andersen was able to use these Toeplitz operators to derive the asymptotic faithfulness of the quantum representations of the mapping class group [1], as well as other results [2, 3, 4].