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DERIVED GROTHENDIECK-TEICHMÜLLER GROUP AND GRAPH COMPLEXES [after T. Willwacher]

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INTRODUCTION

The goal of my talk is to explain the result of Thomas Willwacher (see [**11**]) which relates two universal groups of symmetries introduced in early 90s.

The first symmetry was defined by V. Drinfeld (called Grothendieck-Teichmüller group, or GT in short) as the group of automorphisms of the pro-nilpotent completion of the tower of braid groups (or, more precisely, braid groupoids). An element of the Lie algebra of the graded version GRT of GT is a Lie polynomial in two variables satisfying a complicated set of constraints. Group GT is closely related to the motivic Galois group of $\mathbb{Z}$, and it also appears in a great variety of purely algebraic questions. It acts on various formality morphisms, associators, etc. Multiple-zeta values correspond to functions on a torsor over the GT.

The second group (or more precisely, a differential graded Lie algebra) was defined under the name of "graph complex" in my work on universal symmetries of topological field theories of Chern-Simons type. A cohomology class in graph complex is represented by a finite linear combination of isomorphism classes of finite graphs endowed with an additional data called orientation. There are two graph complexes (even and odd) which differs by the notion of an orientation, depending on the parity of the dimension of the space-time manifold. Graph complexes acts as outer derivations (in the derived sense) of Lie superalgebras of functions on even/odd symplectic supermanifolds, and on rational homotopy types of spaces of embeddings of higher-dimensional spheres.

The relation between GRT and the even graph complex was expected for a long time, as both symmetries act e.g., on the formality morphism in deformation quantization, see e.g., [**9**]. Still, it was not clear how to relate these two theories based on quite different combinatorial structures. In his breakthrough work T. Willwacher found a way to connect them using a generalization of some construction of mine which appeared in the proof of formality of little disks operads. This is a very powerful result, implying non-trivial vanishing results in certain range of parameters (cohomological degree and the weight) on both sides of the story.

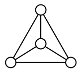
I will be concentrated in my talk on the calculational aspects, and not on various situations where GRT or graph complex act. The original proof from **[11]** is very involved, and contains several dozens of spectral sequences. I will explain the key argument of a streamlined shorter proof extracted from **[5]**.

The first section serves as a short introduction to the basic notions of theory of operads, and of deformation theory. In the second Section I will explain why the derived symmetry group of Gerstenhaber operad could be considered as a derived version of Drinfeld's group GRT. (Even) graph complexes and their relatives appear in the third section. The forth section contains the proof of the most essential part of Willwacher's theorem, and some remarks and open questions.

Finally, I finish the introduction with an example of universal symmetry, which appeared first with wrong coefficients in **[9]**, and was recently corrected in **[1]**.

Example: $\zeta(3)$ flow on Poisson structures

The simplest non-trivial element of the Lie algebra of GRT (dual to $\zeta(3) = \sum_{n\geq 1} \frac{1}{n^3}$) is represented in the graph complex by the complete graph K_4 on 4 vertices:



It produces an evolution equation (see Section 3.3) on the space of bivector fields $\alpha = \sum_{ij} \alpha^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^i}$ in an open domain $\mathcal{U}$ in the coordinate space $\mathbb{R}^N$, given by

$$\frac{d}{dt}\alpha^{i_1 i_2} = \sum_{j_\bullet} \left(\partial_{j_1 j_2 j_3}\alpha^{i_1 i_2}\partial_{j_5}\alpha^{j_1 j_4}\partial_{j_6}\alpha^{j_2 j_5}\partial_{j_4}\alpha^{j_3 j_6} + 6\,\partial_{j_1 j_2}\alpha^{i_1 j_3}\partial_{j_4 j_5}\alpha^{j_1 i_2}\partial_{j_6}\alpha^{j_2 j_4}\partial_{j_3}\alpha^{j_5 j_6}\right)$$

where t is the time variable, $\alpha^{ij} = -\alpha^{ji}$ are C^∞ functions in $\mathcal{U}\times\mathbb{R}_{time}$ (components of tensor α depending on t); the summation is over 6 indices $1 \leq j_1, j_2, j_3, j_4, j_5, j_6 \leq N$.

The solution of this equation with any initial data $\alpha|_{t=0} \in \Gamma(\mathcal{U}, \bigwedge^2 T_{\mathbb{R}^N})$ exists and unique as a *formal power series* in t (or as an analytic germ if the initial data $\alpha|_{t=0}$ is real-analytic). We claim that this equation has the following properties:

1. if $\alpha|_{t=0}$ is a Poisson structure (i.e., the bracket $\{f,g\}_\alpha := \sum_{i_1 i_2} \alpha^{i_1 i_2}\partial_{i_1} f \partial_{i_2} g$ satisfies Jacobi identity), then this property holds for *all* t, understood in the sense of formal series in t (i.e., we get a Poisson bracket on $\mathcal{U}$ with coefficients in $\mathbb{R}[[t]]$),

2. Let $\alpha(t), \beta(t)$ be two Poisson structures depending on time t and obeying the above evolution equation. If at $t = 0$ bivector field α is obtained from β by a change of coordinates, then the same holds for any t, again understood in the sense of formal power series in t.

The conclusion is that we obtain a (formal) flow on isomorphism classes of germs of Poisson structures. Up to now, in all known examples this flow on isomorphism classes is constant. Nevertheless, one can show that it is *impossible* to prove the triviality of the flow using only the universal tensor calculus involving Taylor coefficients of α.

1. TOOLKIT: OPERADS, ALGEBRAS, DEFORMATIONS

1.1. Operads and algebras

Let $\mathbf{k}$ be a field of characteristic zero, denote by $\mathrm{Vect}_{\mathbf{k}}$ the tensor category of vector spaces over $\mathbf{k}$. Any collection $\mathscr{P} = (\mathscr{P}(n))_{n\geq 0}$ of $\mathbf{k}$-linear representations of symmetric groups $(S_n)_{n\geq 0}$ defines a polynomial endofunctor $\Phi_{\mathscr{P}}$ of the category $\mathrm{Vect}_{\mathbf{k}}$ via

$$\Phi_{\mathscr{P}}(V) := \bigoplus_{n\geq 0} \left(\mathscr{P}(n) \otimes V^{\otimes n}\right)_{S_n} \qquad \forall\, V \in \mathrm{Ob}\,(\mathrm{Vect}_{\mathbf{k}}).$$

Obviously, polynomial endofunctors are closed under composition, hence form a monoidal category.

Definition 1.1. — *An* operad $\mathscr{P}$ *over* $\mathbf{k}$ *is a monoid in the monoidal category of polynomial endofunctors. An* algebra *over an operad is an algebra over the corresponding monad in* $\mathrm{Vect}_{\mathbf{k}}$.

Unwinding the definition, one sees immediately that the structure of an operad on a collection $(\mathscr{P}(n))_{n\geq 0}$ of S_n-modules is uniquely determined by the identity element

$$\mathrm{id}_{\mathscr{P}} \in \mathscr{P}(1) \Longleftrightarrow \text{ a morphism } \mathbf{1}_{\mathrm{Vect}_{\mathbf{k}}} \to \mathscr{P}(1)$$

and composition morphisms

$$\mathscr{P}(k) \otimes (\mathscr{P}(n_1) \otimes \mathscr{P}(n_k)) \to \mathscr{P}(n_1 + \cdots + n_k)$$

satisfying certain compatibility constraints. Similarly, the structure of a $\mathscr{P}$-algebra on vector space $V \in \mathrm{Ob}(\mathrm{Vect}_{\mathbf{k}})$ is determined by the collection of S_n-equivariant maps

$$\mathscr{P}(n) \otimes V^{\otimes n} \to V.$$

For any operad $\mathscr{P}$ and any integer $n \geq 0$ the underlying vector space of S_n-representation $\mathscr{P}(n)$ coincides with the space of all natural transformations $A^{\otimes n} \to A$ defined universally for all $\mathscr{P}$-algebras A. For any vector space V the free $\mathscr{P}$-algebra generated by V coincides as a vector space with $\Phi_{\mathscr{P}}(V)$. Basic examples of operads:

– operads $\mathsf{Comm}, \mathsf{Assoc}, \mathsf{Lie}, \mathsf{Poisson}$ describing respectively (non)-unital commutative associative, just associative, Lie or Poisson algebras,
– for any unital associative algebra A define an operad $\mathscr{P}_A$ by $\mathscr{P}_A(1) := A$, $\mathscr{P}_A(n) := 0$ for any $n \neq 0$; the category of $\mathscr{P}_A$-algebras coincides with the category of left A-modules,
– for any vector space V define operad $\mathfrak{End}_V$ by declaring $\mathfrak{End}_V(n) := \mathrm{Hom}(V^{\otimes n}, V)$ for any $n \geq 0$; for any operad $\mathscr{P}$ a structure of a $\mathscr{P}$-algebra on V is the same as a morphism of operads $\mathscr{P} \to \mathfrak{End}_V$.

If $\mathscr{P}$ is one of classical operads $\mathsf{Comm}, \mathsf{Assoc}, \mathsf{Lie}, \mathsf{Poisson}$ then we have $\mathscr{P}(0) = 0$ (because we encode *non-unital* algebras), $\mathscr{P}(1) = \mathbf{k} = \mathbf{1}_{\mathrm{Vect}_{\mathbf{k}}}$ and all operations are generated by binary ones (i.e., by $\mathscr{P}(2)$) subject to appropriate bilinear relations. One has for any $n \geq 1$ the following formula for the dimension:

$$\dim \mathsf{Comm}(n) = 1,\ \dim \mathsf{Assoc}(n) = \dim \mathsf{Poisson}(n) = n!, \dim \mathsf{Lie}(n) = (n-1)!$$

Spaces $\mathscr{P}(n)$ for $\mathscr{P} = \mathsf{Comm}, \mathsf{Assoc}$ or Lie are spanned by operations

$$a_1 \otimes \cdots \otimes a_n \mapsto \begin{cases} a_1 a_2 \ldots a_n & \text{if } \mathscr{P} = \mathsf{Comm}, \\ a_{\sigma(1)} a_{\sigma(2)} \ldots a_{\sigma(n)}, \sigma \in S_n & \text{if } \mathscr{P} = \mathsf{Assoc}, \\ [a_{\sigma(1)}, [a_{\sigma(2)}, [\ldots, [a_{\sigma(n-1)}, a_n] \ldots], \sigma \in S_{n-1} & \text{if } \mathscr{P} = \mathsf{Lie}. \end{cases}$$

Also, one has the following equivalence of polynomial endofunctors:

$$\Phi_{\mathsf{Poisson}} \simeq \Phi_{\mathsf{Comm}} \circ \Phi_{\mathsf{Lie}} \simeq \Phi_{\mathsf{Assoc}}.$$

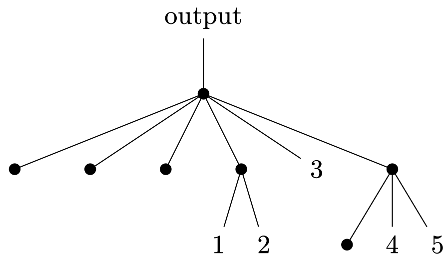
The first equivalence means that universal expression in Poisson algebras are products of universal Lie expressions. The second equivalence follows from the fact that the free unital associative algebra generated by a vector space V is the same as the universal enveloping algebra for the free Lie algebra $\mathfrak{g} = \Phi_{\mathsf{Lie}}(V)$ generated by V, and hence is naturally isomorphic (as a vector space) to the symmetric algebra generated by $\mathfrak{g}$, via Poincaré-Birkhoff-Witt isomorphism $U\mathfrak{g} \simeq Sym\,\mathfrak{g} = \mathbf{1}_{\mathrm{Vect}_{\mathbf{k}}} \oplus \Phi_{\mathsf{Comm}}(V)$.

Remark 1.2 (**Colored operads**). — There is a natural generalization of the language of operads and algebras to the case when algebraic structures under consideration consist not of one, but several distinct vector spaces. One can say that these spaces are "colored" by a set of colors. For example, there is a colored operad with two colors $Algebra, Module$ such that algebras over this colored operad are the same as pairs (A, M) where A is an associative algebra over $\mathbf{k}$ and M is a left A-module. If I denotes the set of colors, then the components of an I-colored operad $\mathscr{P}$ are vector spaces $\mathscr{P}(i_1, \ldots, i_n; j),\ n \geq 0,\ i_1, \ldots, i_n, j \in I$, encoding operations $V_{i_1} \otimes \ldots V_{i_n} \to V_j$, where $(V_i)_{i \in I}$ are colored components of an algebra over colored operad $\mathscr{P}$.

In this way the category of (one-colored) operads can be realized itself as the category of algebras over certain colored operad, whose set of colors is $\mathbb{N} = \{0, 1, 2, \ldots\}$.

The space of color $n \in \mathbb{N}$ is the n-th component $\mathscr{P}(n)$ of a (1-colored) operad $\mathscr{P}$. More conveniently, using representation theory of symmetric groups in zero characteristic, one can make an alternative description with the set of colors corresponding to finite partitions.

We will use later the latter description, for which the category of colored vector spaces is canonically equivalent to the category of collections $\mathscr{P} = (\mathscr{P}(n))_{n\geq 0}$ of $\mathbf{k}$-linear representations of symmetric groups $(S_n)_{n\geq 0}$ (or, in other words, to the category of polynomial endofunctors of $\mathrm{Vect}_{\mathbf{k}}$). The one-colored operads will be algebras of certain partitions-colored operad Oper. If $\mathscr{P} = (\mathscr{P}(n))_{n\geq 0}$ is a collection of (S_n)-representations, then the free operad $\Phi_{\mathsf{Oper}}(\mathscr{P})$ generated by $\mathscr{P}$ is given by the direct sum over certain types of trees of tensor products of components of $\mathscr{P}$, factorized by the automorphism group of the tree. Instead of giving a formal description I will show a typical example from which the general structure should be clear. Consider the following tree:



The corresponding term in the formula is

$$\Phi_{\mathsf{Oper}}(\mathscr{P})(5) = \cdots \oplus \left(\mathscr{P}(0)^{\otimes 4} \otimes \mathscr{P}(2) \otimes \mathscr{P}(3) \otimes \mathscr{P}(6)\right)_{S_3} \oplus \cdots .$$

1.2. Operads in other symmetric monoidal categories

Unwinding the definition of a (colored) operad (and an algebra over an operad) one can easily see that it makes sense in arbitrary symmetric monoidal category instead of $\mathrm{Vect}_{\mathbf{k}}$. For us there will be important three such categories:

- Top: topological spaces (with the tensor product given by the usual product),
- $\mathrm{Vect}_{\mathbf{k}}^{\mathbb{Z}}$: $\mathbb{Z}$-graded vector spaces,
- $\mathrm{Comp}_{\mathbf{k}}$: $\mathbb{Z}$-graded complexes of vector spaces (with the differential of degree $+1$).

In the last two cases the commutativity morphisms $\mathscr{E}^\bullet \otimes \mathscr{F}^\bullet \to \mathscr{F}^\bullet \otimes \mathscr{E}^\bullet$ are, as usual, twisted by the Koszul rule of signs. For any topological operad $\mathscr{P} = (\mathscr{P}(n))_{n\geq 0}$ the collection of chain complexes $(\mathrm{Chains}_\bullet(\mathscr{P}(n), \mathbf{k}))_{n\geq 0}$ is a dg operad (i.e., an operad in $\mathrm{Comp}_{\mathbf{k}}$) concentrated in degrees ≤ 0, and its (co)homology is an operad in $\mathrm{Vect}_{\mathbf{k}}^{\mathbb{Z}}$. Also, one can associate with a dg operad another $\mathbb{Z}$-graded operad just by forgetting the differential. Similarly, any $\mathbb{Z}$-graded operad gives tautologically a dg operad with