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**POISSON ENSEMBLES
OF LOOPS
OF ONE-DIMENSIONAL
DIFFUSIONS**

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Titus LUPU

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**POISSON ENSEMBLES OF LOOPS
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POISSON ENSEMBLES OF LOOPS OF ONE-DIMENSIONAL DIFFUSIONS

Titus Lupu

Abstract. – There is a natural measure on loops (time-parametrized trajectories that in the end return to the origin), which one can associate to a wide class of Markov processes. The Poisson ensembles of Markov loops are Poisson point processes with intensity proportional to these measures. In wide generality, these Poisson ensembles of Markov loops are related, at intensity parameter $1/2$, to the Gaussian free field, and at intensity parameter 1 , to the loops done by a Markovian sample path. Here, we study the specific case when the Markov process is a one-dimensional diffusion. After a detailed description of the measure, we study the Poisson point processes of loops, their occupation fields, and explain how to sample these Poisson ensembles of loops out of diffusion sample path perturbed at their successive minima. Finally, we introduce a couple of interwoven determinantal point processes on the line, which is a dual through Wilson’s algorithm of Poisson ensembles of loops, and study the properties of these determinantal point processes.

Résumé (Ensembles poissonniens de boucles des diffusions unidimensionnelles)

Il y a une mesure naturelle sur les boucles (trajectoires paramétrées par le temps, qui à la fin retournent à leur origine) qu’on peut associer à une large classe de processus de Markov. Les ensembles poissonniens de boucles markoviennes sont des processus ponctuels de Poisson d’intensité proportionnelle à ces mesures. Dans une grande généralité, ces ensembles poissonniens de boucles markoviennes sont reliés, au paramètre d’intensité $1/2$, au champ libre gaussien, et au paramètre d’intensité 1 , aux boucles créées par une trajectoire markovienne. Ici nous étudions le cas spécifique où le processus de Markov est une diffusion unidimensionnelle. Après une description détaillée de la mesure, nous étudions les processus ponctuels de Poisson des boucles, leurs champs d’occupation et expliquons comment séquencer ces ensembles poissonniens de boucles à partir de trajectoires de diffusions perturbées à leur minima successifs. Enfin, nous introduisons un couple de processus ponctuels déterminantaux sur la droite, entrelacés, qui est un dual, à travers l’algorithme de Wilson, de l’ensemble poissonnien de boucles, et étudions les propriétés de ces processus ponctuels déterminantaux.

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CHAPTER 1

INTRODUCTION

1.1. Measure on loops: history

There is a measure on loops one can naturally associate to a wide class of Markov processes. In general, it is expressed as

$$(1.1.1) \quad \mu(d\gamma) := \int_{t>0} \int_x \mathbb{P}_{x,x}^t(d\gamma) p_t(x, x) m(dx) \frac{dt}{t},$$

where $p_t(x, y)$ are the transition densities with respect to a σ -finite measure $m(dy)$, and $\mathbb{P}_{x,y}^t$ are the bridge probability measures. Here, a loop γ is a path parametrized by continuous time, which at the end returns to the origin ($\gamma(0) = \gamma(T(\gamma))$).

Such a measure first appeared, to my knowledge, in the work of Symanzik [61, 62, 63]. There, studying the ϕ^4 model in dimension d , he expressed the moments, and some other functionals, of such fields in terms of multiple integrals over measures on massive Brownian loops and massive Brownian paths, corresponding to a d -dimensional Brownian motion killed at independent exponential time. He called this loop expansion, and saw it as Euclidean analog of Feynman diagrams in Minkowski space QFT. Later, in the work of Brydges, Frölich and Spencer [9], an analogous measure appeared, but for discrete time random walk loops.

Then, an analogously defined measure in the setting of the two-dimensional Brownian motion appeared in the work of Lawler and Werner [34]. They also considered the Poisson point processes of intensity proportional to this measure on loops, which they called “loop-soups”. The main motivation for studying this object was that the obtained loops, up to rerooting and time-reparametrization, were invariant in law by conformal transformations. This properties were used by Sheffield and Werner in [57] to construct the Conformal Loop Ensemble (CLE) as outer boundaries of clusters in a Brownian loop-soup.

Lawler and Trujillo-Ferreras initiated in [33] the study of discrete time random walk loop-soups. See also a recent survey by Lawler [31]. The measure they used was actually the same that appeared in Brydges, Frölich and Spencer [9]. Le Jan

considered loops parametrized by continuous time, associated to Markov jump process on an electrical network, with intensity following the pattern (1.1.1), rather than the discrete time random walk loops [36]. He also called the object Poisson ensemble of Markov loops rather than loop-soup. We will adopt Le Jan's terminology. If one takes the discrete skeleton of Le Jan's loops, one gets the random walk loops studied in [33].

Taking loops parametrized by continuous time allowed Le Jan to consider the occupation field of Poisson ensembles of Markov loops, that is to say the total time spent by loops on each vertex. Le Jan identified two properties universally satisfied by these Poisson ensembles of Markov loops. The first one, is that at intensity parameter $1/2$, the occupation field has same law as half the square of a Gaussian free field. This is an extension of Dynkin's isomorphism [16, 18, 17], and is closely related to Symanzik's identities [63]. The second universal property is that, at intensity parameter 1, the loops erased during the loop-erasure algorithm applied to a Markovian sample path, are part of a Poisson ensembles of Markov loops. This property has been previously observed in the two-dimensional Brownian setting. See [34], Conjecture 1, and [32], Theorem 7.3. See also [56] for recent developments in dimension three. On an electrical network, Wilson's algorithm [69] samples a uniform spanning tree by running successive loop erased random walks. Le Jan observed that the loops erased during this algorithm give a Poisson ensembles of Markov loops of intensity parameter 1.

Since then, the definition of the measure of loops was extended to a wide class of Markov processes, including some not having transition densities [38, 25, 23]. Actually, one rather considers measures on unrooted loops, seen as parametrized by a circle, where the cut between the start and the endpoint is not identified. What makes such measures on unrooted loops natural is actually the covariance by time change by an inverse of a Continuous Additive Functional (CAF). The time change by an inverse of a CAF transforms a Markov process into an other one, possibly on a smaller state space. The measure on unrooted loops of the time changed process is the pushforward by time-change of the measure on loops of the initial Markov process. This covariance is no longer true at the level of rooted loops. In dimension two for Brownian loops, the covariance by time-change implies the invariance by conformal transformations of the range [34].

1.2. Measure on loops: present work and recent developments

These notes are devoted to the study of loop measures, Poisson ensembles of loops and of an analog of Wilson's algorithm in the setting of one-dimensional diffusions. This is a work done during my Ph.D. under the direction of Le Jan at Université Paris-Sud, Orsay [39]. The aim of these notes is to give a complete and self-contained presentation of the topic in the continuous one-dimensional setting, displaying both