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THE RIEMANN-HILBERT MAPPING FOR $\mathfrak{sl}_2$ SYSTEMS OVER GENUS TWO CURVES

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à Étienne Ghys

Abstract. — We prove in two different ways that the monodromy map from the space of irreducible $\mathfrak{sl}_2$ differential systems on genus two Riemann surfaces, towards the character variety of SL_2 representations of the fundamental group, is a local diffeomorphism. We also show that this is no longer true in the higher genus case. Our work is motivated by a question raised by Étienne Ghys about Margulis' problem: the existence of curves of negative Euler characteristic in compact quotients of $\mathrm{SL}_2(\mathbb{C})$.

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RÉSUMÉ (*L'application de Riemann-Hilbert pour les $\mathfrak{sl}_2$-systèmes sur les courbes de genre deux*). — Nous montrons de deux manières différentes que l'application monodromie, depuis l'espace des $\mathfrak{sl}_2$ systèmes différentiels irréductibles sur les surfaces de Riemann de genre deux, vers la variété de caractères des SL_2 représentations du groupe fondamental, est un difféomorphisme local. Nous montrons aussi que ce n'est plus le cas en genre supérieur. Notre travail est motivé par une question d'Étienne Ghys à propos d'un problème de Margulis : l'existence de courbes de caractéristique d'Euler négative dans les quotients compacts de $\mathrm{SL}_2(\mathbb{C})$.

1. Introduction

Let S be a compact oriented topological surface of genus $g \geq 2$ and $X \in \mathrm{Teich}(S)$ a complex structure on S, i.e. X is a smooth projective curve endowed with the isotopy class of a diffeomorphism $X \to S$. Given a $\mathfrak{sl}_2$ matrix of holomorphic one-forms on X:

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathfrak{sl}_2(\Omega^1(X))$$

we consider the system of differential equations for $Y \in \mathbb{C}^2$:

$$\mathrm{d}Y + AY = 0. \tag{1}$$

A fundamental matrix $B(x)$ at a point $x_0 \in X$ is a two-by-two matrix $B \in \mathrm{SL}_2(\mathcal{O}_{x_0})$ whose columns form a base for the two-dimensional vector space of solutions, i.e. satisfying $\mathrm{d}B + AB = 0$, $\det(B) \equiv 1$. It can be analytically continued as a function $B : \widetilde{X} \to \mathrm{SL}_2(\mathbb{C})$ defined on the universal cover of X which satisfies an equivariance

$$\forall \gamma \in \pi_1(X), \quad B(\gamma \cdot x) = B(x) \cdot \rho_A(\gamma)^{-1}$$

for a certain representation $\rho_A : \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$. The conjugacy class of ρ_A in the $\mathrm{SL}_2(\mathbb{C})$ character variety

$$\Xi := \mathrm{Hom}(\pi_1(S), \mathrm{SL}_2(\mathbb{C}))//\mathrm{SL}_2(\mathbb{C})$$

does not depend on the initial solution and will be referred to as the **monodromy class** of the system. Also, for any $M \in \mathrm{SL}_2(\mathbb{C})$, the monodromy class of $MAM^{-1} \in \mathfrak{sl}_2(\Omega^1(X))$ coincides with that of A. It is therefore natural to consider the space of systems up to gauge equivalence:

$$\mathrm{Syst} := \{(X, A) : X \in \mathrm{Teich}(S), A \in \mathfrak{sl}_2(\Omega^1(X))\}//\mathrm{SL}_2(\mathbb{C}).$$

The Riemann-Hilbert mapping is the map

$$\mathrm{Mon} : \mathrm{Syst} \to \Xi$$

defined by $\mathrm{Mon}(X, [A]) := [\rho_A]$. Both Syst and Ξ are (singular) algebraic varieties of complex dimension $6g - 6$. The irreducible locus $\mathrm{Syst}^{\mathrm{irr}} \subset \mathrm{Syst}$ and

$\Xi^{\mathrm{irr}} \subset \Xi$, characterized by those A and image(ρ) without nontrivial invariant subspace, define smooth open subsets. Then, Mon induces a holomorphic mapping between these open sets. Our main aim is to prove the following:

THEOREM 1.1. — *If S has genus two, then the holomorphic map*

$$\mathrm{Mon} : \mathrm{Syst}^{irr} \to \Xi^{irr}$$

is a local diffeomorphism.

The equivalent statement is not true in general for higher genera. Easy counter-examples in genus at least 4 can be constructed by considering the pull back of a system on a genus two Riemann surface X by a parameterized family of ramified coverings over X. In genus $g = 3$ there are also counter-examples (see Section 6 for details). On the other hand, we note that irreducibility is a necessary assumption. Indeed, diagonal and nilpotent systems

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & -\alpha \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}$$

admit nontrivial isomonodromic deformations: this comes from isoperiodic deformations of pairs (X, α) that exist, as can be seen just by counting dimensions (see also [9]).

Motivation. The question of determining the properties of the monodromy representations associated to holomorphic $\mathfrak{sl}_2$ systems on a Riemann surface X of genus $g > 1$ was raised by Ghys. The motivation comes from the study of quotients $M := \mathrm{SL}_2/\Gamma$ by cocompact lattices $\Gamma \subset \mathrm{SL}_2$. These compact complex manifolds are not Kähler. Huckleberry and Margulis proved in [8] that they admit no complex hypersurfaces (and therefore no nonconstant meromorphic functions). Elliptic curves exist in such quotients, while the existence of compact curves of genus at least two remains open and is related to Ghys' question. Indeed, assuming that for a nontrivial system on a curve X, its monodromy has an image contained in Γ (up to conjugation), then the corresponding fundamental matrix induces a nontrivial holomorphic map from X to M. Reciprocally, any curve X in M can be lifted to $\mathrm{SL}(2, \mathbb{C})$ and gives rise to the fundamental matrix of some system on X, whose monodromy is contained in Γ. In fact, it is not known whether holomorphic $\mathfrak{sl}_2$ systems on Riemann surfaces of genus > 1 give rise to representations with discrete or real images. Although Ghys' question remains open, our result shows on the one hand that we can locally realize arbitrary deformations of the monodromy representation of a given system over a genus two curve by allowing deformations of both the curve and the system, and on the other that if a genus two curve exists in some M as before, it is rigid in M up to left translations.

Idea of the proofs. We propose two different proofs of our result, using isomonodromic deformations of two kinds of objects, namely vector bundles with connections used by the last two authors, and branched projective structures used by the first two authors. Both proofs were obtained independently. We decided to write them together in this paper. We first develop the approach with flat vector bundles in Sections 2, 3 and 4. It is based on the work [7] by the last two authors, where the arguments occur at the level of (a finite covering of) the moduli space of systems. In more detail, one considers a system as a holomorphic $\mathfrak{sl}_2$ connection $\nabla = \mathrm{d} + A$ on a trivial bundle $X \times \mathbb{C}^2 \to X$, and thinks of it as a point in the (larger) moduli space Con of all triples (X, E, ∇) where $E \to X$ is a holomorphic rank two vector bundle and ∇ is a flat $\mathfrak{sl}_2$ connection on E. The subspace Syst of those triples over trivial bundles has codimension three. The monodromy map is locally defined on the larger space Con and its level sets induce a singular foliation $\mathcal{F}_{\mathrm{iso}}$ by three-dimensional leaves: the isomonodromy leaves. We note that everything is smooth in restriction to the irreducible locus, and Mon is a submersion. Since $\mathcal{F}_{\mathrm{iso}}$ and Syst have complementary dimensions, the fact that the restriction of Mon to $\mathrm{Syst}^{\mathrm{irr}}$ is a local diffeomorphism is equivalent to the transversality of $\mathcal{F}_{\mathrm{iso}}$ to $\mathrm{Syst}^{\mathrm{irr}}$. To prove this, we strongly use the hyperellipticity of genus two curves to translate our problem to some moduli space of logarithmic connections on $\mathbb{P}^1$. The main tool here, due to Goldman, is that irreducible $\mathrm{SL}_2(\mathbb{C})$ representations are invariant under the hyperelliptic involution $h : X \to X$, and descend to representations of the orbifold quotient X/h. There, isomonodromy equations are well-known, explicitly given by a Garnier system, and we can compute the transversality.

The approach using branched projective structures is developed in Sections 5, 6, 7, 8, and 9. We hope that it might be generalized to higher genus. It uses isomonodromic deformation spaces of branched complex projective structures over a surface S of genus $g \geq 2$, that were introduced in [2]; namely, given a conjugacy class of irreducible representation $\rho : \pi_1(S) \to \mathrm{SL}(2, \mathbb{C})$, and an even integer k, the space of complex projective structures with k branch points (counted with multiplicity) and holonomy ρ has the structure of a smooth k-dimensional complex manifold denoted by $\mathcal{M}_{k,\rho}$. We establish a dictionary between

a) Systems on $X \in \mathrm{Teich}(S)$ with monodromy ρ
b) Regular holomorphic foliations on $X \times \mathbb{P}^1$ transverse to the $\mathbb{P}^1$-fibration and with monodromy $[\rho] \in \mathrm{Hom}(\pi_1(S), \mathrm{PSL}_2(\mathbb{C}))$ (Riccati foliations)
c) Complete rational curves in the space $\mathcal{M}_{2g-2,\rho}$ of branched projective structures over S with branching divisor of degree $2g - 2$ and monodromy ρ.

The equivalence between (a) and (b) is a simple and well-known fact. The interesting aspect of the dictionary is between (a) and (c), as discussed in Section 6. Injectivity of the differential of the Riemann-Hilbert mapping at a given $\mathfrak{sl}(2)$ system with monodromy ρ is then equivalent to first-order rigidity of the corresponding rational curve in $\mathcal{M}_{2g-2,\rho}$. In the case of genus two, the moduli space $\mathcal{M}_{2,\rho}$ is a complex surface, and the rigidity of the rational curve is established by showing its self-intersection is equal to -4. In higher genus, the infinitesimal rigidity of the rational curve does not hold; counter-examples are described in Section 11.

Finally, in Section 10, we compare the objects involved in the two proofs. A branched projective structure on X can be viewed as triple $(P, \mathcal{F}, \sigma)$ where

- $P \to X$ is a ruled surface (i.e. total space of a $\mathbb{P}^1$-bundle),
- $\mathcal{F}$ is a regular Riccati foliation on P (i.e. transverse to all $\mathbb{P}^1$-fibers),
- $\sigma : X \to P$ is a section which is not $\mathcal{F}$-invariant.

Branch points come from tangencies between the section $\sigma(X)$ and the foliation $\mathcal{F}$. On the other hand, a $\mathrm{SL}(2,\mathbb{C})$-connection (E,∇) defines, after projectivization, a projective connection $\mathbb{P}\nabla$ on the $\mathbb{P}^1$-bundle $\mathbb{P}E$ whose horizontal sections are the leaves of a Riccati foliation $\mathcal{F}$ on the total space P of the bundle. In this setting, a section $\sigma : X \to P$ inducing a projective structure with $2g-2$ branch points corresponds to a line sub-bundle $L \subset E$ having degree 0, implying in particular that E is strictly semi-stable. We use this dictionary in Section 10 to explain how isomonodromic deformations considered in each proof are related. This allows us to have a better understanding of the geometry of isomonodromic deformations of $\mathrm{SL}(2,\mathbb{C})$-connections around points in the locus Syst, where E is a trivial bundle.

2. $\mathfrak{sl}_2$ systems on X

Let us consider the smooth projective curve of genus two defined in an affine chart by

$$X_{\boldsymbol{t}} := \{y^2 = x(x-1)(x-t_1)(x-t_2)(x-t_3)\} \tag{2}$$

for some parameter

$$\boldsymbol{t} = (t_1, t_2, t_3) \in T := \left(\mathbb{P}^1 \setminus \{0,1,\infty\}\right)^3 \setminus \bigcup_{i \neq j} \{t_i = t_j\}. \tag{3}$$

The space T is a finite covering of the moduli space of genus two curves. A $\mathfrak{sl}_2$ system on $X_{\boldsymbol{t}}$ takes the form

$$\mathrm{d}Y + AY = 0 \quad \text{with} \quad A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \tag{4}$$

where α, β, γ are holomorphic 1-forms on $X_{\boldsymbol{t}}$. It can be seen as the equation $\nabla Y = 0$ for ∇-horizontal sections for the $\mathfrak{sl}_2$ connection $\nabla = \mathrm{d} + A$ on the trivial bundle over $X_{\boldsymbol{t}}$.