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SKELETONS AND MODULI OF STOKES TORSORS

BY JEAN-BAPTISTE TEYSSIER

ABSTRACT. – We prove an analog for Stokes torsors of Deligne's skeleton conjecture and deduce from it the representability of the functor of relative Stokes torsors by an affine scheme of finite type over C. This provides, in characteristic 0, a local analog of the existence of a coarse moduli for skeletons with bounded ramification, due to Deligne. As an application, we use the geometry of this moduli to derive quite strong finiteness results for integrable systems of differential equations in several variables which did not have any analog in one variable.

RÉSUMÉ. – Nous prouvons une variante pour les torseurs de Stokes de la conjecture des squelettes de Deligne, et en déduisons la représentabilité du foncteur des torseurs de Stokes relatifs par un schéma affine de type fini sur C. Cela fournit, en caractéristique 0, un analogue local de l'existence d'un espace de modules grossier pour les squelettes à ramification bornée, dû à Deligne. À titre d'application, nous utilisons la géométrie de cet espace de modules pour en déduire de nouveaux résultats de finitude sur les systèmes intégrables d'équations différentielles à plusieurs variables qui n'avaient pas d'analogue à une variable.

Consider the following linear differential equation (E) with polynomial coefficients:

$$p_n \frac{d^n f}{dz^n} + p_{n-1}\frac{d^{n-1} f}{dz^{n-1}} + \cdots + p_1 \frac{df}{dz} + p_0 f = 0.$$

If $p_n(0) \neq 0$, Cauchy theorem asserts that a holomorphic solution to (E) defined on a small disk around 0 is equivalent to the values of its n first derivatives at 0. If $p_n(0) = 0$, holomorphic solutions to (E) on a small disk around 0 may always be zero. Nonetheless, (E) may have formal power series solutions. The *Main asymptotic development theorem* [26], due to Hukuhara and Turrittin asserts that for a direction θ emanating from 0, and for a formal power series solution f, there is a sector $\mathcal{S}_\theta$ containing θ such that f can be "lifted" in a certain sense to a holomorphic solution f_θ of (E) on $\mathcal{S}_\theta$. We say that f_θ *is asymptotic to* f *at* 0. If f_θ is analytically continued around 0 into a solution $\widetilde{f_\theta}$ of (E) on the sector $\mathcal{S}_{\theta'}$, where $\theta' \neq \theta$, it may be that the asymptotic development of $\widetilde{f_\theta}$ at 0 is not f any more. This is the *Stokes phenomenon*. As a general principle, the study of (E) amounts to the study of its "formal type" and the study of how asymptotic developments of solutions jump via

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analytic continuation around 0. To organize these informations, it is traditional to adopt a linear algebra point of view.

The equation (E) can be seen as a *differential module*, i.e., a finite dimensional vector space $\mathcal{N}$ over the field $\mathbb{C}\{z\}[z^{-1}]$ of convergent Laurent series, endowed with a $\mathbb{C}$-linear endomorphism $\nabla : \mathcal{N} \longrightarrow \mathcal{N}$ satisfying the Leibniz rule. In this language, solutions of (E) correspond to elements of Ker ∇ (also called *flat sections of* ∇). Furthermore, a differential equation with same "formal type" as (E) corresponds to a differential module $\mathcal{M}$ with an isomorphism of *formal* differential modules iso : $\mathcal{M}_{\widehat{0}} \longrightarrow \mathcal{N}_{\widehat{0}}$. Since iso can be seen as a formal flat section of the differential module $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$, the main asymptotic development theorem applies to it. The lifts of iso to sectors thus produce a cocycle $\gamma := (\mathrm{iso}_\theta \, \mathrm{iso}_{\theta'}^{-1})_{\theta,\theta' \in S^1}$ with value into the sheaf of sectorial automorphisms of $\mathcal{N}$ which are asymptotic to Id at 0. This is the *Stokes sheaf* of $\mathcal{N}$, denoted by $\mathrm{St}_{\mathcal{N}}(\mathbb{C})$. A fundamental result of Malgrange [17] and Sibuya [25] implies that $(\mathcal{M}, \mathrm{iso})$ is determined by the torsor under $\mathrm{St}_{\mathcal{N}}(\mathbb{C})$ associated to the cocycle γ. Hence, *Stokes torsors encode in an algebraic way analytic data and classifying differential equations amounts to studying Stokes torsors*. As a result, the study of Stokes torsors is meaningful.

In higher dimension, the role played by differential modules is played by *good meromorphic connections*. We will take such a connection $\mathcal{N}$ defined around $0 \in \mathbb{C}^n$ to be of the shape

$$\mathcal{E}^{a_1} \otimes \mathcal{R}_{a_1} \oplus \cdots \oplus \mathcal{E}^{a_d} \otimes \mathcal{R}_{a_d}, \tag{0.0.1}$$

where the a_i are meromorphic functions with poles contained in a normal crossing divisor D, where $\mathcal{E}^{a_i}$ stands for the rank one connection $(\mathcal{O}_{\mathbb{C}^n,0}(*D), d - da_i)$, and where the $\mathcal{R}_{a_i}$ are regular connections. Note that from works of Kedlaya [12, 13] and Mochizuki [20, 22], every meromorphic connection is (up to ramification) formally isomorphic at each point to a connection of the form (0.0.1) at the cost of blowing-up enough the pole locus. If $(r_i, \theta_i)_{i=1,\dots,n}$ are the usual polar coordinates on $\mathbb{C}^n$, the Stokes sheaf $\mathrm{St}_{\mathcal{N}}$ of $\mathcal{N}$ is a sheaf of complex unipotent algebraic groups over the torus $T := (S^1)^n$ defined by $r_1 = \cdots = r_n = 0$.

By a *$\mathcal{N}$-marked connection*, we mean the data $(\mathcal{M}, \nabla, \mathrm{iso})$ of a meromorphic connection $(\mathcal{M}, \nabla)$ around 0 endowed with an isomorphism of formal connections at the origin iso : $\mathcal{M}_{\widehat{0}} \longrightarrow \mathcal{N}_{\widehat{0}}$. As in dimension 1, Mochizuki [21, 22] showed that $\mathcal{N}$-marked connections are determined by their associated Stokes torsor, so we consider them as elements in $H^1(T, \mathrm{St}_{\mathcal{N}}(\mathbb{C}))$.

Since $\mathrm{St}_{\mathcal{N}}$ is a sheaf of complex algebraic groups, its sheaf of R-points $\mathrm{St}_{\mathcal{N}}(R)$ is a well-defined sheaf of groups on T for any commutative $\mathbb{C}$-algebra R. Consequently, one can consider the functor of relative Stokes torsors $R \longrightarrow H^1(T, \mathrm{St}_{\mathcal{N}}(R))$, denoted by $H^1(T, \mathrm{St}_{\mathcal{N}})$. Following a strategy designed by Deligne, Babbitt and Varadarajan [2] proved that $H^1(S^1, \mathrm{St}_{\mathcal{N}})$ is representable by an affine space. Hence in dimension 1, the set of torsors under $\mathrm{St}_{\mathcal{N}}$ has a structure of a complex algebraic variety.

The interest of this result is to *provide a framework in which questions related to differential equations can be treated with the apparatus of algebraic geometry*. This might look like a wish rather than a documented fact since the local theory of linear differential equations is fully understood in dimension 1 by means of analysis. In dimension ≥ 2 however, new phenomena appear and this geometric perspective seems relevant. As we will show, the representability of $H^1(T, \mathrm{St}_{\mathcal{N}})$ in any dimension implies for differential equations quite strong finiteness results which have no counterparts in dimension 1 and which seem out of reach with former technology. Thus, we prove the following:

THEOREM 1. – *The functor $H^1(T, \mathrm{St}_{\mathcal{N}})$ is representable by an affine scheme of finite type over C.*

Before explaining how the proof relates to Deligne's skeleton conjecture, let us describe two applications to finiteness results.

Suppose that $\mathcal{N}$ is *very good*, that is, for functions a_i, a_j appearing in (0.0.1) with $a_i \neq a_j$, the difference $a_i - a_j$ has poles along all the components of the divisor D along which $\mathcal{N}$ is localized. Let V be a manifold containing 0 and let us denote by $\mathcal{N}_V$ the restriction of the connection $\mathcal{N}$ to V. We prove the following:

THEOREM 2. – *If V is transverse to every irreducible component of D, there is only a finite number of equivalence classes of $\mathcal{N}$-marked connections with given restriction to V. Furthermore, this number depends only on $\mathcal{N}$ and on V.*

This theorem looks like a weak differential version of Lefschetz's theorem. A differential Lefschetz theorem would assert that for a generic choice of V, $\mathcal{N}$-marked connections are determined by their restriction to V. It is a hope of the author that such a question is approachable by geometric means using the morphism of schemes

$$\mathrm{res}_V : H^1(T, \mathrm{St}_{\mathcal{N}}) \longrightarrow H^1(T', \mathrm{St}_{\mathcal{N}_V}) \tag{0.0.2}$$

induced by the restriction to V.

To give flesh to this intuition, let us indicate how geometry enters the proof of Theorem 2. Since unramified morphisms of finite type have finite fibers, it is enough to show that $\mathcal{N}$-marked connections lie in the unramified locus of res_V, which is the locus where the tangent map of res_V is injective. We show in 5.2 a canonical identification

$$T_{(\mathcal{M},\nabla,\mathrm{iso})} H^1(T, \mathrm{St}_{\mathcal{N}}) \simeq \mathcal{H}^1(\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_0, \tag{0.0.3}$$

where the left-hand side denotes the tangent space of $H^1(T, \mathrm{St}_{\mathcal{N}})$ at $(\mathcal{M}, \nabla, \mathrm{iso})$ and where $\mathcal{H}^1\,\mathrm{Sol}$ denotes the first cohomology sheaf of the solution complex of a $\mathcal{D}$-module. Note that the left-hand side of (0.0.3) is *algebraic*, whereas the right-hand side is *transcendental*. From (0.0.3) we deduce a similar transcendental interpretation for the kernel $\mathrm{Ker}\, T_{(\mathcal{M},\nabla,\mathrm{iso})}\, \mathrm{res}_V$ and prove its vanishing using a perversity theorem due to Mebkhout [18].

Using an invariance theorem due to Sabbah [24], we further prove the following rigidity result:

THEOREM 3. – *Suppose that D has at least two components and that $\mathcal{N}$ is very general. Then there is only a finite number of equivalence classes of $\mathcal{N}$-marked connections.*

In this statement, very general means that $\mathcal{N}$ is very good and that the residues of each regular constituent contributing to $\mathcal{N}$ in (0.0.1) lie away from a countable union of strict Zariski closed subsets of the affine space.

Let us finally explain roughly the proof of Theorem 1. The main idea is to import and prove a conjecture from the field of Galois representations. Let X be a smooth variety over a finite field of characteristic $p > 0$, and let $\ell \neq p$ be a prime number. To any ℓ-adic local system $\mathcal{F}$ on X up to semi-simplification, one can associate its *skeleton* $\mathrm{sk}\,\mathcal{F}$, that is