ISSN 0012-9593

quatrième série - tome 52 fascicule 2 mars-avril 2019

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Keerthi MADAPUSI PERA

Toroidal Compactifications of Integral Models of Shimura Varieties of Hodge Type

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
annales@ens.fr

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : abonnements@smf.emath.fr

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4^e série, t. 52, 2019, p. 393 à 514

TOROIDAL COMPACTIFICATIONS OF INTEGRAL MODELS OF SHIMURA VARIETIES OF HODGE TYPE

BY KEERTHI MADAPUSI PERA

ABSTRACT. – We construct toroidal compactifications for integral models of Shimura varieties of Hodge type. We also construct integral models of the minimal (Satake-Baily-Borel) compactification. Our results essentially reduce the problem to understanding the integral models themselves. As such, they cover all previously known cases of PEL type. At primes where the level is hyperspecial, we show that our compactifications are canonical in a precise sense. We also provide a new proof of Y. Morita's conjecture on the everywhere good reduction of abelian varieties whose Mumford-Tate group is anisotropic modulo center. Along the way, we demonstrate an interesting rationality property of Hodge cycles on abelian varieties with respect to p-adic analytic uniformizations.

RÉSUMÉ. – Nous construisons des compactifications toroïdales pour les modèles entiers de variétés de Shimura de type de Hodge. Nous construisons également la compactification minimale (ou de Satake-Baily-Borel) pour ces modèles entiers. Nos résultats réduisent le problème à la compréhension des modèles entiers eux-mêmes. Donc ils recouvrent tous les cas déjá connus de type PEL. Quand le niveau est hyperspécial, nous montrons que nos compactifications sont canoniques dans un sens précis. Nous fournissons une nouvelle preuve de la conjecture de Y. Morita sur la bonne réduction de variétés abéliennes dont le groupe de Mumford-Tate est anisotrope modulo son centre. Sur le chemin, nous démontrons une propriété de rationalité intéressante de cycles de Hodge sur les variétés abéliennes par rapport aux uniformisations analytiques p-adiques.

Introduction

Shimura varieties of Hodge type. – This paper is concerned with constructing compactifications for integral models of Shimura varieties of Hodge type. Essentially, these are the Shimura varieties that can be viewed as parameter spaces for polarized abelian varieties equipped with level structures and additional Hodge tensors.

More precisely, we will work with Shimura data (G, X) that admit embeddings into a Siegel Shimura datum $(\mathrm{GSp}(V), \mathrm{S}^{\pm}(V))$ attached to a symplectic space V over $\mathbb{Q}$. Given such an embedding and a small enough compact open $K \subset G(\mathbb{A}_f)$, we have the associated Shimura variety $\mathrm{Sh}_K(G, X)$, which has the above moduli interpretation over $\mathbb{C}$.

 doi:10.24033/asens.2391

If we are in the more familiar PEL setting, the additional Hodge tensors parameterized by $\mathrm{Sh}_K(G,X)(\mathbb{C})$ can be chosen to consist of endomorphisms and polarizations. One can then define representable PEL type moduli problems over the reflex field $E = E(G,X)$, and even over a suitable localization of its ring of integers, which recover the moduli interpretation for $\mathrm{Sh}_K(G,X)$ over $\mathbb{C}$, and are thus *canonical* models for $\mathrm{Sh}_K(G,X)$ over E or even its ring of integers; cf. [14] for the theory over E, and [34] for the integral theory (away from primes where the level is not hyperspecial).

The theory of [14] applies more generally to show that Shimura varieties of Hodge type admit canonical models over their reflex fields (1), and Milne has used Deligne's results on absolute Hodge cycles to give these canonical models a modular interpretation; cf. [47].

Example. – An important class of Shimura data of Hodge type arises from quadratic forms over $\mathbb{Q}$ of signature $(n+,2-)$. Suppose that we have a vector space U over $\mathbb{Q}$ equipped with such a quadratic form. Then the group $G = \mathrm{GSpin}(U)$ acts naturally on the Clifford algebra C attached to U. We can equip C with an appropriate symplectic form such that we have an embedding $\mathrm{GSpin}(U) \hookrightarrow \mathrm{GSp}(C)$. Moreover, if we take X to be the space of negative definite oriented 2-planes in $U_{\mathbb{R}}$, then (G,X) is a Shimura datum, and we in fact get an embedding $(G,X) \hookrightarrow (\mathrm{GSp}(C), \mathrm{S}^{\pm})$ of Shimura data. This is the *Kuga-Satake construction*; cf. [15]. So (G,X) is of Hodge type, but is not of PEL type when $n \geq 6$.

The Shimura varieties attached to such data are important, for example, in the study of the moduli of K3 surfaces (when $n = 19$). Moreover, the Shimura varieties attached to the Spin group Shimura data play a significant role in S. Kudla's program (cf. [35]) for relating intersection numbers on Shimura varieties with Fourier coefficients of Eisenstein series. They have also been used by the author to prove the Tate conjecture for K3 surfaces; cf. [42].

Integral models. – Unfortunately, since Hodge cycles are transcendentally defined, there is no natural way to use them to obtain a modular interpretation over the ring of integers of E. But an *ad hoc* construction of integral models can be given as follows: fix a prime p and a place $v|p$ of E. Suppose that we have an embedding $(G,X) \hookrightarrow (\mathrm{GSp}, \mathrm{S}^{\pm})$ into a Siegel Shimura datum. For any level $K^{\ddagger} \subset \mathrm{GSp}(\mathbb{A}_f)$, the Siegel Shimura variety $\mathrm{Sh}_{K^{\ddagger}}(\mathrm{GSp}, \mathrm{S}^{\pm})$ has a natural integral model $\mathcal{S}_{K^{\ddagger}}$ over $\mathbb{Z}_{(p)}$: this is Mumford's construction.

We can now take the normalization of $\mathcal{S}_{K^{\ddagger}}$ in $\mathrm{Sh}_K(G,X)$ (2): This gives us a normal integral model $\mathcal{S}_K$ over $\mathcal{O}_{E,(v)}$, which is finite over $\mathcal{S}_{K^{\ddagger}}$.

When $p > 2$ and the level at p is hyperspecial, Kisin showed in [32] that $\mathcal{S}_K$ is a smooth scheme $\mathcal{O}_{E,(v)}$, and is canonical in a precise sense. (3) In particular, it is independent of the choice of symplectic embedding.

In general, however, one does not know if $\mathcal{S}_K$ has any good properties. Moreover, it need not be independent of the choice of symplectic embedding.

(1) We now know that every Shimura variety admits such a canonical model; cf. [46].
(2) Here, and in the rest of the paper, given a normal, excellent $\mathbb{Z}_{(p)}$-algebraic stack S, an open dense substack $j : U \hookrightarrow S$, and a finite map $f : S' \to U$, with S' normal, the *normalization of S in S'* will be the finite S-algebraic stack, whose associated coherent sheaf of $\mathcal{O}_S$-algebras is the normalization of $\mathcal{O}_S$ in $j_* f_* \mathcal{O}_{S'}$.
(3) A related result due to Vasiu can be found in [60]. The result was also extended to the case $p = 2$ in [31].

Compactifications. – In any case, since our interest lies mainly in the computation of the zeta function of the Shimura variety, and hence its cohomology, we are led to consider the question of compactifying $\mathscr{S}_K$.[4]

Another motivation to study compactifications of integral models is the role they play in constructing regular proper models over $\mathbb{Z}$ for the orthogonal Shimura varieties mentioned above. Such models are a crucial ingredient in carrying out Kudla's program on the arithmetic intersection theory of these spaces; cf. [43] for a description of these models over $\mathbb{Z}[1/2]$.

Over $\mathbb{C}$, Mumford and his collaborators (cf. [3]) constructed good, toroidal compactifications in the general setting of arithmetic quotients of hermitian symmetric domains. In [24] and [56] these compactifications are constructed for Shimura varieties in their natural adélic setting. All these constructions depend on a choice of a certain cone decomposition Σ, called a complete admissible rpcd (cf. § 4 for the terminology). Given such a choice they produce a compactification Sh_K^{Σ} of the Shimura variety $\mathrm{Sh}_K := \mathrm{Sh}_K(G, X)$ with good properties.

In the Siegel case, when the level $K^{\ddagger}$ is hyperspecial at p, Chai and Faltings [20] studied degenerations of abelian varieties, and used this to construct smooth compactifications $\mathscr{S}_{K^{\ddagger}}^{\Sigma'}$ of $\mathscr{S}_{K^{\ddagger}}$ over $\mathbb{Z}_{(p)}$ attached to smooth cone decompositions Σ' for the symplectic group. It was shown by K.-W. Lan [37] that this construction is compatible in characteristic 0 with the analytic construction of Mumford, *et. al.* mentioned above. Lan's proof uses a careful comparison of the algebraic and analytic definitions of theta functions. We give an independent proof of this fact here, using the compatibility of Mumford's construction with cohomological realizations; cf. (2.2.13).

For the general Hodge type case, there is a natural cone decomposition Σ attached to (G, X) such that the normalization of $\mathrm{Sh}_{K^{\ddagger}}^{\Sigma'}$ in Sh_K is canonically isomorphic to the toroidal compactification Sh_K^{Σ}; cf. [24].

Now, assume that $K^{\ddagger}$ is chosen to be hyperspecial at p (this can always be arranged using Zarhin's trick, and by replacing the Siegel Shimura data with one associated with a larger symplectic space). If we take the normalization of $\mathscr{S}_{K^{\ddagger}}^{\Sigma^{\ddagger}}$ in Sh_K, we obtain a proper normal algebraic space $\mathscr{S}_K^{\Sigma}$ over $\mathscr{O}_{E,(v)}$, whose generic fiber is Sh_K^{Σ}, and which contains $\mathscr{S}_K$ as an open sub-scheme.

The main result of this paper is:

THEOREM 1. – *$\mathscr{S}_K^{\Sigma}$ is a compactification of $\mathscr{S}_K$. More precisely, the complement of $\mathscr{S}_K$ in $\mathscr{S}_K^{\Sigma}$ is a relative Cartier divisor over $\mathscr{O}_{E,(v)}$. Moreover, $\mathscr{S}_K^{\Sigma}$ admits a stratification of the expected shape, extending that of its generic fiber. After replacing Σ by an appropriate refinement if necessary, the singularities of $\mathscr{S}_K^{\Sigma}$ are no worse than those of $\mathscr{S}_K$: Every complete local ring of $\mathscr{S}_K^{\Sigma}$ at a geometric point is isomorphic to a complete local ring of $\mathscr{S}_K$.*[5]

For the reader familiar with the general structure of toroidal compactifications, we can unpack Theorem 1 a bit (cf. 4.1.5): $\mathscr{S}_K^{\Sigma}$ is stratified by locally closed sub-schemes, and each stratum in this stratification can be described as follows: There is a normal integral model $\mathscr{S}_{K_{\Phi,h}}(G_{\Phi,h}, D_{\Phi,h})$ over $\mathscr{O}_{E,(v)}$ of a Shimura variety, a projective scheme

(4) We will see below that this question is largely independent of the properties of the integral model itself.

(5) The original version of this theorem imposed the further condition that the level subgroup at p satisfies $K_p = K_p^{\ddagger} \cap G(\mathbb{Q}_p)$. We thank the referee for encouraging us to consider the situation at arbitrary level.