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ERGODIC PROPERTIES OF SOME NEGATIVELY CURVED MANIFOLDS WITH INFINITE MEASURE

Pierre Vidotto

Abstract. – Let $\mathrm{M} = \mathrm{X}/\Gamma$ be a geometrically finite negatively curved manifold with fundamental group Γ acting on X by isometries. The purpose of this book is to study the mixing property of the geodesic flow on $\mathrm{T}^1\mathrm{M}$, the asymptotic behavior as $R \longrightarrow +\infty$ of the number of closed geodesics on M of length less than R and of the orbital counting function $\#\{\gamma \in \Gamma \mid \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}$.

These properties are well known when the Bowen-Margulis measure on $\mathrm{T}^1\mathrm{M}$ is finite. We consider here Schottky group $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$ whose Bowen-Margulis measure is infinite and ergodic, such that one of the elementary factor Γ_i is parabolic with $\delta_{\Gamma_i} = \delta_\Gamma$. We specify these ergodic properties using a symbolic coding induced by the Schottky group structure.

Résumé. – Soit $\mathrm{M} = \mathrm{X}/\Gamma$ une variété géométriquement finie de courbure strictement négative et Γ son groupe fondamental agissant par isométries sur X. Nous étudions successivement dans cet article une propriété de mélange du flot géodésique sur $\mathrm{T}^1\mathrm{M}$, le comportement quand $R \longrightarrow +\infty$ du nombre de géodésiques fermées de M de longueur plus petite que R et celui de la fonction orbitale $\#\{\gamma \in \Gamma \mid \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}$.

Ces propriétés sont bien connues dans le cas où la mesure de Bowen-Margulis est finie sur $\mathrm{T}^1\mathrm{M}$. Nous considérons ici un groupe de Schottky $\Gamma = \Gamma_1 * \Gamma_2 * \cdots * \Gamma_k$ de mesure de Bowen-Margulis infinie et ergodique, pour lequel au moins un facteur Γ_i est parabolique et satisfait $\delta_{\Gamma_i} = \delta_\Gamma$. Les propriétés ergodiques ci-dessus sont alors précisées, en utilisant un codage symbolique induit par la structure de groupe de Schottky de Γ.

CONTENTS

CHAPTER 1

INTRODUCTION

1.1. Background and previous results

Let X be a connected, simply connected and complete riemannian manifold with pinched negative sectional curvature. Denote by d the distance on X induced by the riemannian structure of X and by Γ a discrete group of isometries of (X, d), acting properly discontinuously without fixed point and let $\mathrm{M} = \mathrm{X}/\Gamma$. Fix $\mathbf{o} \in \mathrm{X}$. The study of quantities like the orbital function

$$\mathrm{N}_\Gamma(\mathbf{o}, R) := \#\{\gamma \in \Gamma \mid \mathrm{d}(\mathbf{o}, \gamma \cdot \mathbf{o}) \leqslant R\}$$

is strongly related to the dynamics of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on the unit tangent bundle $\mathrm{T}^1\mathrm{M}$ of the quotient manifold. Let us first define precisely this flow: each pair $(\mathbf{p}, \mathbf{v}) \in \mathrm{T}^1\mathrm{M}$ determines a unique geodesic $(\boldsymbol{\gamma}(t))_{t\in\mathbb{R}}$ satisfying $(\boldsymbol{\gamma}(0), \boldsymbol{\gamma}'(0)) = (\mathbf{p}, \mathbf{v})$ and for any $t \in \mathbb{R}$, the action of g_t is given by $g_t(\mathbf{p}, \mathbf{v}) = (\boldsymbol{\gamma}(t), \boldsymbol{\gamma}'(t))$. It is known (see **[37]**) that the topological entropy of the geodesic flow is given by the rate of exponential growth δ_Γ of the orbital function, that is

$$\delta_\Gamma := \limsup_{R\longrightarrow+\infty} \frac{\ln\left(\mathrm{N}_\Gamma(\mathbf{o}, R)\right)}{R}.$$

This last quantity is also the critical exponent of the *Poincaré series* $\mathscr{P}_\Gamma$ of the group Γ defined as follows: for any $s > 0$

$$\mathscr{P}_\Gamma(s) := \sum_{\gamma\in\Gamma} e^{-s\mathrm{d}(\mathbf{o}, \gamma\cdot\mathbf{o})}.$$

S. J. Patterson (in **[39]**) and D. Sullivan (in **[43]**) used these series to construct a family of measures $(\sigma_{\mathbf{x}})_{\mathbf{x}\in\mathrm{X}}$, the so-called Patterson-Sullivan measures. More precisely, each measure $\sigma_{\mathbf{x}}$ is fully-supported by the limit set $\Lambda_\Gamma \subset \partial\mathrm{X}$, which is defined as the set of all accumulation points of one (all) Γ-orbit(s) in the visual boundary $\partial\mathrm{X}$ of X. This set is also the smallest non-empty Γ-invariant closed subset of $\mathrm{X} \cup \partial\mathrm{X}$. It is the closure in the boundary of the set of fixed points of $\Gamma^* := \Gamma\backslash\{\mathrm{Id}\}$. A group Γ is said to be elementary if its limit set is a finite set. S.J. Patterson and D. Sullivan described a process to associate to this family a measure m_Γ defined on $\mathrm{T}^1\mathrm{M}$, which is invariant under the action of the geodesic flow. When the group Γ is *divergent*, i.e., $\mathscr{P}_\Gamma(\delta_\Gamma) = +\infty$ (otherwise Γ is said to be *convergent*), the family $(\sigma_{\mathbf{x}})_{\mathbf{x}\in\mathrm{X}}$ is unique up to a normalization, hence m_Γ is also unique. We will focus in this book on the case of divergent groups, which allows us to speak about "the" Bowen-Margulis measure m_Γ even when m_Γ has infinite mass. Nevertheless, in this introduction, the assumption "$\mathrm{M} = \mathrm{X}/\Gamma$ has infinite Bowen-Margulis measure" should be in general understood as the fact that *any* invariant measure obtained from a Patterson-Sullivan density $(\sigma_{\mathbf{x}})_{\mathbf{x}\in\mathrm{X}}$ has infinite mass. We first study here a property of mixing of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ with respect to this measure. We say that the geodesic flow $(g_t)_{t\in\mathbb{R}}$ is mixing with respect to a measure m with finite total mass $||m||$ on $\mathrm{T}^1\mathrm{M}$, if for any m-measurable sets $\mathfrak{A}, \mathfrak{B} \subset \mathrm{T}^1\mathrm{M}$, one gets

$$m\left(\mathfrak{A} \cap g_{-t}\mathfrak{B}\right) \longrightarrow \frac{m(\mathfrak{A})m(\mathfrak{B})}{||m||} \text{ as } t \longrightarrow \pm\infty. \tag{1}$$

When the measure m has infinite mass, this definition may be extended saying that the flow $(g_t)_{t\in\mathbb{R}}$ is mixing if

$$m\left(\mathfrak{A} \cap g_{-t}\mathfrak{B}\right) \longrightarrow 0 \text{ as } t \longrightarrow \pm\infty, \text{ where } \mathfrak{A} \text{ and } \mathfrak{B} \text{ have finite measure.}$$

When the measure m_Γ is finite, Property (1) was first proved by G. A. Hedlund in **[27]** for finite volume surfaces in constant curvature, by F. Dal'bo and M. Peigné for Schottky groups with parabolic isometries acting on Hadamard manifolds with pinched negative curvature (see **[17]**) and by M. Babillot in the general case (see **[1]**). The following result of T. Roblin **[41]** gathers all the information known in such a general content.

THEOREM (Roblin). – *If the Bowen-Margulis measure m_Γ has finite mass $||m_\Gamma||$ (resp. infinite mass), the flow $(g_t)_{t\in\mathbb{R}}$ satisfies*

$$m_\Gamma\left(\mathfrak{A} \cap g_{-t}\mathfrak{B}\right) \underset{t\longrightarrow\pm\infty}{\longrightarrow} \frac{m_\Gamma(\mathfrak{A})m_\Gamma(\mathfrak{B})}{||m_\Gamma||} \left(\textit{resp. } m_\Gamma\left(\mathfrak{A} \cap g_{-t}\mathfrak{B}\right) \underset{t\longrightarrow\pm\infty}{\longrightarrow} 0\right).$$

REMARK 1.1.1. – *The definition of mixing in infinite measure seems to be weak (see the third chapter of* **[41]** *about this fact). Nevertheless, our Theorem A below will furnish an asymptotic of the form*

$$m_\Gamma(\mathfrak{A} \cap g_{-t}\mathfrak{B}) \underset{t\longrightarrow\pm\infty}{\sim} \varepsilon(|t|)m_\Gamma(\mathfrak{A})m_\Gamma(\mathfrak{B}), \textit{ for an explicit function } \varepsilon,$$

which can be understood as a mixing property, up to a renormalization.

On the one hand, this property is interesting from the point of view of the ergodic theory. On the other hand, in the case of geometrically finite manifolds with finite measure, the property of mixing of the geodesic flow may be used to find an asymptotic of the orbital function $\mathrm{N}_\Gamma(\mathbf{o}, R)$. This idea was initially developed in G.A. Margulis'