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NIP, KEISLER MEASURES AND COMBINATORICS [after S. Shelah, H.J. Keisler, E. Hrushovski, Y. Peterzil, A. Pillay, P. Simon, ...]

by **Sergei STARCHENKO**

INTRODUCTION

Keisler measures were introduced by H.J. Keisler in [**20**] as finitely additive probability measures on Boolean algebras of definable sets. A deep insight of H.J. Keisler was that many ideas and tools of stability theory can be extended to so-called NIP theories by replacing types (i.e., 0-1 valued measures) by arbitrary probability measures.

Almost 20 years later Keisler's work was revisited, significantly improved and deepened in a series of papers by E. Hrushovski, Y. Peterzil, A. Pillay, S. Shelah, P. Simon and others (e.g., see [**30, 31, 17, 18, 19**]). Probability measures played an essential role in a proof of Pillay's conjecture for o-minimal groups ([**17**]), Hrushovski's work on approximate subgroups ([**15, 8**]) and understanding topological dynamics in NIP structures ([**6**]).

Recently it was observed that Keisler measures in distal theories provide a natural framework for certain problems in combinatorics and allow one to generalize some Ramsey-type results from the semi-algebraic case to a wider class of fields (e.g., p-adics) and also to so-called generically stable measures. (See Theorems 4.2 and 4.4 below.)

To illustrate the role of distality consider the following consequence of Theorem 4.2, that we call the Points-Lines Property.

Points-Lines Property. *There is $\delta > 0$ such that for a large enough finite set of points $P \subseteq \mathbb{R}^2$ and a large enough finite set of lines L in $\mathbb{R}^2$ of the form $y = ax + b$ there are $P_0 \subseteq P$, $L_0 \subseteq L$ with $|P_0| \geq \delta|P|$, $L_0| \geq \delta|P|$ and $p \notin l$ for any $p \in P_0$, $l \in L_0$.*

Moreover there are semi-algebraic families $\mathscr{F} \subseteq \mathscr{P}(\mathbb{R}^2)$ and $\mathscr{G} \subseteq \mathscr{P}(\mathbb{R}^2)$, independent of P and L, such that $P_0 = P \cap F$ for some $F \in \mathscr{F}$ and $L_0 = L \cap G$ for some $G \in \mathscr{G}$ (here we identify $(a, b) \in \mathbb{R}^2$ with the line $y = ax + b$).

In a sense, the field of real numbers is optimal for results like the Points-Lines Property. Of course, identifying, as usual, the complex plane $\mathbb{C}^2$ with $\mathbb{R}^4$ and using Theorem 4.2, we obtain that the Points-Lines Property holds for points and lines in the complex plane. However, first of all we don't know of any proof for the field $\mathbb{C}$ that would not involve, in one form or another, real algebraic geometry. Secondly, in the moreover part we cannot replace "semi-algebraic" by "algebraic" (i.e., definable in the field of complex numbers). Also, to some surprise, the Points-Lines Property fails in any algebraically closed field of positive characteristic, even without the moreover part, (see [**7**, Proposition 6.2]). Model theoretically, an explanation for why the field of real numbers is more suited for the above type results is distality: the field $\mathbb{R}$ is distal, while no algebraically closed field is distal (see Theorems 4.4 and 4.10 for a relation between Ramsey-type results and distality).

In this paper we will present basics on NIP, Keisler measures, distality and also demonstrate their use in combinatorics.

For an understanding of a basic theory of Keisler measures some knowledge of model theory is needed. In Section 1 we will provide a very informal introduction to basic model theoretic notions and explain them in more details in the cases of algebraically closed and real closed fields that we will use throughout the paper. We refer to the book [**33**] for more details on NIP and Keisler measures.

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1. MODEL THEORETIC PRELIMINARIES

In this section we give a short informal introduction to some model theoretic notions such as structures, formulas, definable sets, etc. that we will use in the paper. More details can be found in any introductory model theory book (e.g., [**25, 35**]).

A first order structure (or just *a structure*) $\mathscr{M}$ is a non-empty set M (called the universe of $\mathscr{M}$) together with a set of distinguished (also called basic) functions, relations and constants. If $f \colon M^n \to M$ is a distinguished function then we refer to n as the arity of f.

For example the field of complex numbers can be viewed as a structure with the universe $\mathbb{C}$ equipped with addition, multiplication, the function $z \mapsto -z$ and two constants 0 and 1.