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THE HODGE THEORY OF THE DECOMPOSITION THEOREM [after M. A. de Cataldo and L. Migliorini]

by **Geordie WILLIAMSON**

INTRODUCTION

The Decomposition Theorem is a beautiful theorem about algebraic maps. In the words of MacPherson [**25**], "it contains as special cases the deepest homological properties of algebraic maps that we know." Since its proof in 1981 it has found spectacular applications in number theory, representation theory and combinatorics. Like its cousin the Hard Lefschetz Theorem, proofs appealing to the Decomposition Theorem are usually difficult to obtain via other means. This leads one to regard the Decomposition Theorem as a deep statement lying at the heart of diverse problems.

The Decomposition Theorem was first proved by Beilinson, Bernstein, Deligne and Gabber [**2**]. Their proof proceeds by reduction to positive characteristic in order to use the Frobenius endomorphism and its weights, and ultimately rests on Deligne's proof of the Weil conjectures. Some years later Saito obtained another proof of the Decomposition Theorem as a corollary of his theory of mixed Hodge modules [**29, 28**]. Again the key is a notion of weight.

More recently, de Cataldo and Migliorini discovered a simpler proof of the Decomposition Theorem [**4, 6**]. The proof is an ingenious reduction to statements about the cohomology of smooth projective varieties, which they establish via Hodge theory. In their proof they uncover several remarkable geometric statements which go a long way to explaining "why" the Decomposition Theorem holds, purely in the context of the topology of algebraic varieties. For example, their approach proves that the intersection cohomology of a projective variety is of a motivic nature ("André motivated") [**3**]. Their techniques were adapted by Elias and the author to prove the existence of Hodge theories attached to Coxeter systems ("Soergel modules"), thus proving the Kazhdan-Lusztig positivity conjecture [**18**].

The goal of this article is to provide an overview of the main ideas involved in de Cataldo and Migliorini's proof. A striking aspect of the proof is that it gathers the Decomposition Theorem together with several other statements generalising the Hard Lefschetz Theorem and the Hodge-Riemann Bilinear Relations (the "Decomposition Theorem Package"). Each ingredient is indispensable in the induction. One is left with the impression that the Decomposition Theorem is not a theorem by itself, but rather belongs to a family of statements, each of which sustains the others.

Before stating the Decomposition Theorem we recall two earlier theorems concerning the topology of algebraic maps. The first (Deligne's Degeneration Theorem) is an instance of the Decomposition Theorem. The second (Grauert's Theorem) provides an illustration of the appearance of a definite form, which eventually forms part of the "Decomposition Theorem Package".

0.1. Deligne's Degeneration Theorem

Let $f : X \to Y$ be a smooth (i.e., submersive) projective morphism of complex algebraic varieties. Deligne's theorem asserts that the Leray spectral sequence

$$E_2^{pq} = H^p(Y, R^q f_* \mathbb{Q}_X) \Rightarrow H^{p+q}(X, \mathbb{Q}) \tag{1}$$

is degenerate (i.e., $E_2 = E_\infty$). Of course such a statement is false for submersions between manifolds (e.g., the Hopf fibration). The theorem asserts that something very special happens for smooth algebraic maps.

Let us recall how one may construct the Leray spectral sequence. In order to compute the cohomology of X we replace the constant sheaf $\mathbb{Q}_X$ on X by an injective resolution. Its direct image on Y then has a natural "truncation" filtration whose successive subquotients are the (shifted) higher direct image sheaves $R^q f_* \mathbb{Q}_X[-q]$. This filtered complex of sheaves gives rise to the Leray spectral sequence.

In fact, Deligne proved that there exists a decomposition in the derived category of sheaves on Y:

$$Rf_* \mathbb{Q}_X \cong \bigoplus_{q \geq 0} R^q f_* \mathbb{Q}_X[-q] \tag{2}$$

(i.e., the filtration of the previous paragraph splits). The decomposition in (2) implies the degeneration of (1), and in fact is the universal explanation for such a degeneration. Deligne also proved that each local system $R^q f_* \mathbb{Q}_X$ is semi-simple. Hence the object $Rf_* \mathbb{Q}_X$ is as semi-simple as we could possibly hope. This is the essence of the Decomposition Theorem, as we will see.

Because $f : X \to Y$ is smooth and projective any fibre of f is a smooth projective variety. Deligne deduces the decomposition in (2) by applying the Hard Lefschetz Theorem to the cohomology of the fibres of f. Thus the decomposition of $Rf_* \mathbb{Q}_X$ is