

409

ASTÉRISQUE

2019

A TORSION JACQUET-LANGLANDS CORRESPONDENCE

Frank CALEGARI & Akshay VENKATESH

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Astérisque est un périodique de la Société Mathématique de France.

Numéro 409, 2019

Comité de rédaction

Marie-Claude ARNAUD Fanny KASSEL
Christophe BREUIL Alexandru OANCEA
Damien CALAQUE Nicolas RESSAYRE
Philippe EYSSIDIEUX Sylvia SERFATY
Nicolas BURQ (dir.)

Diffusion

Maison de la SMF AMS
Case 916 - Luminy P.O. Box 6248
13288 Marseille Cedex 9 Providence RI 02940
France USA
commandes@smf.emath.fr <http://www.ams.org>

Tarifs

Vente au numéro: 45 € (\$ 67)

Abonnement Europe: 665 €, hors Europe: 718 € (\$ 1 077)

Des conditions spéciales sont accordées aux membres de la SMF.

Secrétariat

Astérisque
Société Mathématique de France
Institut Henri Poincaré, 11, rue Pierre et Marie Curie
75231 Paris Cedex 05, France
Fax: (33) 01 40 46 90 96
asterisque@smf.emath.fr • <http://smf.emath.fr/>

© Société Mathématique de France 2019

Tous droits réservés (article L 122-4 du Code de la propriété intellectuelle). Toute représentation ou reproduction intégrale ou partielle faite sans le consentement de l'éditeur est illicite. Cette représentation ou reproduction par quelque procédé que ce soit constituerait une contrefaçon sanctionnée par les articles L 335-2 et suivants du CPI.

ISSN: 0303-1179 (print) 2492-5926 (electronic)
ISBN 978-2-85629-903-6
doi:10.24033/ast.1075

Directeur de la publication: Stéphane Seuret

409

ASTÉRISQUE

2019

A TORSION JACQUET-LANGLANDS CORRESPONDENCE

Frank CALEGARI & Akshay VENKATESH

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Frank Calegari
Department of Mathematics,
University of Chicago,
5734 S University Ave,
Chicago, IL 60637, USA

Akshay Venkatesh
School of Mathematics,
Institute for Advanced Study,
1 Einstein Drive,
Princeton, New Jersey 08540, USA

Mathematical Subject Classification (2010). — 11F75, 11F80, 11F70, 57M27.

Mots-clefs. — Représentations de Galois, cohomologie des groupes arithmétiques, programme de Langlands.

Keywords. — Galois Representations, Cohomology of Arithmetic Groups, Langlands Program.

A TORSION JACQUET-LANGLANDS CORRESPONDENCE

par Frank CALEGARI & Akshay VENKATESH

Abstract. — We prove a numerical form of a Jacquet-Langlands correspondence for torsion classes on arithmetic hyperbolic 3-manifolds.

Résumé. (Une correspondance de Jacquet-Langlands torsion) — Nous établissons une forme numérique d'une correspondance de Jacquet-Langlands pour les classes de torsion sur des variétés hyperboliques arithmétiques de dimension 3.

CONTENTS

Acknowledgments	ix
1. Introduction	1
1.1. Introduction	1
1.2. A guide to reading this book	6
2. Some Background and Motivation	9
2.1. Reciprocity over \mathbf{Z}	9
2.2. Inner forms of $GL(2)$: conjectures	14
3. Notation	21
3.1. A summary of important notation	21
3.2. Fields and adèles	25
3.3. The hyperbolic 3-manifolds	27
3.4. Homology, cohomology, and spaces of modular forms	29
3.5. Normalization of metric and measures	34
3.6. S -arithmetic groups	35
3.7. Congruence homology	37
3.8. Eisenstein classes	45
3.9. Automorphic representations. Cohomological representations	46
3.10. Newforms and the level raising/level lowering complexes	47
4. Raising the Level: newforms and oldforms	51
4.1. Ihara's lemma	51
4.2. No newforms in characteristic zero.	56
4.3. Level raising	60
4.4. The spectral sequence computing the cohomology of S -arithmetic groups	62
4.5. $\zeta(-1)$ and the homology of PGL_2	67
5. The split case	81
5.1. Noncompact hyperbolic manifolds: height functions and homology ..	82
5.2. Noncompact hyperbolic manifolds: eigenfunctions and Eisenstein series	86
5.3. Reidemeister and analytic torsion	92
5.4. Noncompact arithmetic manifolds	98

5.5. Some results from Chapter 4 in the split case	102
5.6. Eisenstein series for arithmetic manifolds: explicit scattering matrices	106
5.7. Modular symbols, boundary torsion, and the Eisenstein regulator ...	113
5.8. Comparing Reidmeister and analytic torsion: the main theorems ...	125
5.9. Small eigenvalues	130
5.10. The proof of Theorem 5.8.3	155
6. Comparisons between Jacquet-Langlands pairs	163
6.1. Notation	163
6.2. The classical Jacquet Langlands correspondence	164
6.3. Newforms, new homology, new torsion, new regulator	164
6.4. Torsion Jacquet-Langlands, crudest form	166
6.5. Comparison of regulators and level-lowering congruences: a conjecture	167
6.6. Torsion Jacquet-Langlands, crude form: matching volume and congruence homology	174
6.7. Essential homology and the torsion quotient	176
6.8. Torsion Jacquet-Langlands, refined form: spaces of newforms	180
6.9. The general case	187
7. Numerical examples	191
7.1. The manifolds	192
7.2. No characteristic zero forms	193
7.3. Characteristic zero oldforms	194
7.4. Characteristic zero newforms and level lowering	197
7.5. Eisenstein Deformations: Theoretical Analysis	204
7.6. Eisenstein Deformations: Numerical Examples	208
7.7. Phantom classes	209
7.8. $K_2(\mathcal{O}_F)$ and $F = \mathbf{Q}(\sqrt{-491})$	213
7.9. Table	215
Bibliography	217
Index	225

ACKNOWLEDGMENTS

The first author (F.C.) would like to thank Matthew Emerton for many conversations regarding possible integral formulations of reciprocity, and to thank Nathan Dunfield; the original calculations suggesting that a torsion Jacquet-Langlands theorem might be true were done during the process of writing our joint paper [17], and the data thus produced proved very useful for suggesting some of the phenomena we have studied in this book. He would also like to thank Kevin Hutchinson for some helpful remarks concerning a theorem of Suslin. F.C. was supported during the preparation of this book by a Sloan fellowship and by the National Science Foundation (CAREER Grant DMS-0846285, NSF Grant DMS-1404620, NSF Grant DMS-1648702, and NSF Grant DMS-1701703).

The second author (A.V.) would like to express his gratitude to Nicolas Bergeron, with whom he wrote the paper [7], for many fruitful discussions and thoughts about the analytic behavior of analytic torsion; to Laurent Clozel, who suggested to him the importance of investigating torsion in cohomology; to Kartik Prasanna, for helpful discussions and references concerning his work; and to Avner Ash for encouraging words. He was supported during the preparing of this book by a Sloan fellowship, by National Science Foundation grants DMS-1065807 and DMS-1401622, and a David and Lucile Packard Foundation fellowship; he gratefully thanks these organizations for their support.

The mathematical debt this book owes to the work of Cheeger [24] and Müller [68] should be clear: the main result about comparison of torsion homology makes essential use of their theorem.

Some of the original ideas of this book were conceived during the conference “Explicit Methods in Number Theory” in Oberwolfach during the summer of 2007. Various parts of the manuscript were written while the authors were resident at the following institutions: New York University, Stanford University, Northwestern University, the Institute for Advanced Study, the University of Chicago, the University of Melbourne, the University of Sydney, and the Brothers K coffee shop; we thank all of these institutions for their hospitality.

We thank Toby Gee, Florian Herzig, Krzysztof Klosin, Kai-Wen Lan, and Romyar Sharifi for helpful comments on a previous version of this manuscript. We thank Jean Raimbault and Gunter Harder for pointing out errors in Chapter 5. We thank