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WEAKLY CURVED A_∞-ALGEBRAS OVER A TOPOLOGICAL LOCAL RING

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Abstract. – We define and study the derived categories of the first kind for curved DG- and A_∞-algebras complete over a pro-Artinian local ring with the curvature elements divisible by the maximal ideal of the local ring. We develop the Koszul duality theory in this setting and deduce the generalizations of the conventional results about A_∞-modules to the weakly curved case. The formalism of contramodules and comodules over pro-Artinian topological rings is used throughout the memoir. Our motivation comes from the Floer-Fukaya theory.

Résumé. – Nous définissons et étudions les catégories dérivées de la première espèce pour les dg-algèbres et les A_∞-algèbres à courbure sur un anneau pro-Artinien local où les éléments de courbure sont divisibles par l'idéal maximal de l'anneau local. Nous développons la théorie de la dualité de Koszul dans ce cadre et déduisons des généralisations au cas de la courbure faible des résultats classiques sur les A_∞-modules. Dans tout ce mémoire, nous nous servons systématiquement du formalisme des contramodules et comodules sur un anneau topologique pro-Artinien. Notre motivation vient de la théorie de Floer-Fukaya.

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CHAPTER 0

INTRODUCTION

0.1. – The conventional definition of the derived category involves localizing the homotopy category (or alternatively, the category of complexes and closed morphisms between them) by the class of quasi-isomorphisms. In fact, one needs more than just such a definition in order to do homological algebra in a derived category. Constructing derived functors and computing the Ext groups requires having appropriate classes of resolutions.

The *classical homological algebra* can be roughly described as the study of derived categories that can be equivalently defined as the localizations of the homotopy categories by quasi-isomorphisms and the full subcategories in the homotopy categories formed by complexes of projective or injective objects, or DG-modules that are projective/injective as graded modules.

This is true, e. g., for appropriately bounded complexes over an abelian or exact category with enough projectives or injectives, or for bounded DG-modules over a DG-algebra with nonpositive cohomological grading, or over a connected simply connected DG-algebra with nonnegative cohomological grading.

0.2. – It is known since the pioneering work of Spaltenstein [**49**] that one can work with the unbounded derived categories of modules and sheaves using resolutions satisfying stronger conditions than termwise projectivity, flatness or injectivity. These classes of resolutions, now known as *homotopy projective*, *homotopy flat*, etc., complexes, are defined by conditions imposed on the complex as a whole and depending on the differential in the complex, rather than only on its terms. This approach was extended to unbounded DG-modules over unbounded DG-rings by Keller [**27**] and Bernstein-Lunts [**3**].

Another point of view, first introduced in Hinich's paper [**22**] in the case of cocommutative DG-coalgebras, involves strengthening the conditions imposed on quasi-isomophisms rather than the conditions on resolutions. Extended to DG-comodules by Lefèvre-Hasegawa [**31**] and others, this theory took its fully developed form in the present author's monograph **[41]** and memoir **[42]**, where the general definitions of the *derived categories of the second kind* were given.

The terminology of *two kinds of derived categories* goes back to the classical paper of Husemoller, Moore, and Stasheff **[23]**, where the distinction between *differential derived functors of the first* and *the second kind* was introduced. The conventional unbounded derived category, studied by Spaltenstein, Keller, et al., is called *the derived category of the first kind*. The definition of the derived category of the second kind has several versions, called the *absolute derived*, *complete derived*, *coderived*, and *contraderived category* **[42, 37]**. Here the "coderived category" terminology comes from Keller's brief exposition **[28]** of the related results from Lefèvre-Hasegawa's thesis.

0.3. – As the latter terminological system suggests, the coderived categories are more suitable for comodules, and similarly the contraderived categories more suitable for contramodules **[12, 46]**, than for modules. The philosophy of "taking the derived categories of the first kind for modules, and the derived categories of the second kind for comodules or contramodules" was used in the author's monograph on semi-infinite homological algebra **[41]**. It works well in Koszul duality, too **[42]**, although the derived categories of the second kind for DG-modules also have their uses in the case of DG-algebras whose underlying graded algebras have finite homological dimension.

Unlike the conventional quasi-isomorphism, the equivalence relations used to define the derived categories of the second kind are not reflected by forgetful functors; so one cannot tell whether, say, a DG-comodule is trivial in the coderived category (*coacyclic*) or not just by looking on its underlying complex of vector spaces. Thus the forgetful functors between derived categories of the second kind are generally *not* conservative (which stands in the way of possible application of the presently popular techniques of the ∞-categorical Barr-Beck theorem **[33]**).

On the other hand, derived categories of the second kind make perfect sense for curved DG-modules or DG-comodules **[42]**, to which the conventional definition of the derived category (of the first kind) is not applicable, as curved structures have no cohomology groups. This sometimes forces one to consider derived categories of the second kind for modules, including modules over rings of infinite homological dimension, inspite of all the arising technical complications **[37, 10]**.

The aim of this memoir is to show how the derived category of the first kind can be defined for curved DG-modules and curved A_∞-modules, if only in a rather special situation of algebras over a complete local ring with the curvature element divisible by the maximal ideal of the local ring.

0.4. – Let us explain the distinction between the derived categories of the first and the second kind in some more detail (see also **[42**, Sections 0.1–0.3**]** and **[41**, Preface and Section 0.2.9**]**). The classical situation, when there is no difference between the two kinds of derived categories, is special in that no infinite summation occurs when one totalizes a resolution of a complex or a DG-module. When the need to use the infinite summation arises, however, one is forced to choose between taking direct sums or direct products.

Informally, this means specifying the direction along the diagonals of a bicomplex (or a similar double-indexed family of groups with several differentials) in which the terms "increase" or "decrease" in the order of magnitude. Using the appropriate kind of completion is presumed, in which one takes infinite direct sums in the "increasing" direction and infinite products in the "decreasing" one. The components of the total differential of the bicomplex-like structure are accordingly ordered; and the spectral sequence in which one first passes to the cohomology with respect to the dominating component of the differential converges (at least in the weak sense of [**11**]) to the cohomology of the related totalization.

In the A_∞-algebra situation, the conventional theory of the first kind presumes that the operations m_i, $i \geqslant 1$, are ordered so that $m_1 \gg m_2 \gg m_3 \gg \cdots$ in the order of magnitude, i.e., the component m_1 dominates. So, in particular, if the differential $d = m_1$ is acyclic on a given A_∞-algebra or A_∞-module, then such an algebra or module vanishes in the homotopy category and the higher operations m_2, m_3, etc. on it do not matter from the point of view of the theory of the first kind.

On the other hand, considering the derived category of the second kind, e. g., for CDG-modules over a CDG-algebra, means choosing the multiplication as the dominating component, i.e., setting $m_2 \gg m_1 \gg m_0$. Hence the importance of the underlying graded algebras or modules of CDG-algebras or CDG-modules, with the differentials and the curvature elements in them forgotten, in the study of the coderived, contraderived, and absolute derived categories.

Of course, the above vague wording should be taken with a grain of salt, and the notation is symbolic: it is not any particular maps m_i (some of which may well happen to vanish for some particular algebras) but the whole vector spaces of such maps that are ordered in the "order of magnitude".

0.5. – A theory of the second kind for curved A_∞ structures can also be developed. Essentially, it would mean that no "divergent" infinite sequences of the higher operations should be allowed to occur. As usually, it is technically easier to do that for coalgebras, where the "convergence condition" on the higher comultiplications μ_i appears naturally. This theory is reasonably well-behaved [**42**, Sections 7.4–7.6].

It may be possible to have a theory of the second kind for curved A_∞-algebras, too, e. g., by restricting oneself to those curved A_∞-algebras and A_∞-modules in each of which there is a finite number of nonvanishing higher operations m_i only [**42**, final sentences of Remark 7.3]. However, proving theorems about such A_∞-modules would involve working with topological coalgebras, which seems to be technically quite unpleasant (infinite operations on modules would be problematic, etc.)

A more delicate approach might involve imposing the convergence condition according to which the operations m_i eventually vanish in the restriction to the tensor powers of every fixed finite-dimensional subspace of the curved A_∞-algebra, and similarly for the higher components f_i of the curved A_∞-morphisms, modules, etc. The bar-construction of such an A_∞-algebra would be defined as a DG-coalgebra object in the tensor category of ind-pro-finite-dimensional vector spaces.

Perhaps the most reasonable way to deal with the aforementioned problem would require replacing the ground category of vector spaces with that of pro-vector spaces, with the approximate effect of interchanging the roles of algebras and coalgebras [**41**, Remark 2.7]. This is also what one may wish to do should the need to develop a theory of the first kind for A_∞-coalgebras arise (cf. [**42**, Remark 7.6]).

0.6. – On the other hand, having a theory of the first kind for curved A_∞-algebras would essentially mean setting $m_0 \gg m_1 \gg m_2 \gg \cdots$, i.e., designating the curvature element m_0 as the dominant term. The problem is that m_0, being just an element of a curved A_∞-algebra, is too silly a structure to be allowed to dominate unrestrictedly. When nondegenerate enough (and it is all too easy for an element of a vector space to be nondegenerate enough) and made dominating, it would kill all the other structure of such a curved A_∞-algebra or A_∞-module.

That is why every curved A_∞-algebra over a field which is either considered as nonunital and has a nonzero curvature element, or has a curvature element not proportional to the unit, is A_∞-isomorphic to a curved A_∞-algebra with $m_i = 0$ for all $i \geqslant 1$ [**42**, Remark 7.3]. Moreover, every A_∞-module over a (unital or not) curved A_∞-algebra with a nonzero curvature element over a field is contractible.

0.7. – Hence the alternative of developing a theory of the first kind for curved A_∞-algebras over a local ring $\mathfrak{R}$, with the curvature element being required to be divisible by the maximal ideal $\mathfrak{m} \subset \mathfrak{R}$. Let us first discuss this idea in the simplest case of the ring of formal power series $\mathfrak{R} = k[[\epsilon]]$, where k is a field.

In terms of the above ordering metaphor, this means having two scales of orders of magnitude at the same time. On the one hand, the ϵ-adic topology is presumed, i.e., $1 \gg \epsilon \gg \epsilon^2 \gg \cdots$ On the other hand, it is a theory of the first kind, so $m_0 \gg m_1 \gg m_2 \gg \cdots$ Given that m_0 is assumed to be divisible by ϵ and m_1 isn't, the question which of the two scales has the higher priority arises immediately.

If we want to make our theory as far from trivial as possible, the natural answer is to designate the ϵ-adic scale as the more important one. In the theory developed in this memoir, this is achieved by having the topology of a complete local ring $\mathfrak{R}$ built into the tensor categories of $\mathfrak{R}$-modules in which our A_∞-algebras and A_∞-modules live. That is where $\mathfrak{R}$-contramodules (and also $\mathfrak{R}$-comodules) come into play.

0.8. – In the conventional setting of (uncurved) DG- and A_∞-algebras over a field, the notion of A_∞-morphisms can be used to define the derived category of DG-modules. Indeed, the homotopy category of A_∞-modules over an A_∞-algebra coincides with their derived category, and the derived category of A_∞-modules over a DG-algebra is equivalent to the derived category of DG-modules. So the complex of A_∞-morphisms between two DG-modules over a DG-algebra computes the Hom between them in the derived category of DG-modules.

A similar definition of the derived category of curved DG-modules over a curved DG-algebra was suggested in [**36**]. Then it was shown in the subsequent paper [**29**] that the "derived category of CDG-modules" defined in this way vanishes entirely whenever the curvature element of the CDG-algebra is nonzero and one is working over a field (as it follows from the above discussion).

One of the results of this memoir is the demonstration of a setting in which this kind of definition of the derived category of curved DG-modules is nontrivial and reasonably well-behaved.

0.9. – Before we start explaining what $\mathfrak{R}$-contramodules and $\mathfrak{R}$-comodules are, let us have a look on the situation from another angle.

The passage from uncurved to curved algebras is supposed not only to expand the class of algebras being considered, but also enlarge the sets of morphisms between them. In fact, the natural functor from DG-algebras to CDG-algebras is faithful, but not fully faithful [**40**]. Together with the curvature elements in algebras, *change-of-connection* elements in morphisms between algebras are naturally supposed to come.

One of the consequences of the existence of the change-of-connection morphisms in the category of CDG-algebras is the impossibility of extending to CDG-modules the conventional definition of the derived category (of the first kind) of DG-modules over a DG-algebra. Quite simply, CDG-isomorphic DG-algebras may have entirely different derived categories of DG-modules. Moreover, the derived category of DG-modules over a DG-algebra is invariant under quasi-isomorphisms of DG-algebras; and this already is incompatible with the functoriality with respect to change-of-connection morphisms. Indeed, *any* two DG-algebras over a field can be connected by a chain of transformations, some of which are quasi-isomorphisms, while the other ones are CDG-isomorphisms of DG-algebras [**42**, Examples 9.4].

0.10. – So another problem with the naïve attempt to develop a theory of the first kind for curved A_∞-algebras over a field is that one cannot have change-of-connection morphisms in it. One can say that the A_∞-morphisms between such A_∞-algebras are too numerous, in that all the operations $m_1, m_2, \ldots$ can be killed by A_∞-isomorphisms if only the curvature element m_0 is not proportional to the unit, and still they are too few, in that morphisms with nonvanishing change-of-connection components f_0 cannot be considered.

To be more precise, recall that an A_∞*-morphism* $f\colon A \longrightarrow B$ is defined a sequence of maps $f_i\colon A^{\otimes i} \longrightarrow B$, where $i \geqslant 1$ [**31**]. For curved A_∞-algebras, one would like to define curved A_∞-morphisms as similar sequences of maps f_i starting with $i = 0$, the component $f_0 \in B^1$ being the change-of-connection element. The problem is that the compatibility equations on the maps f_i in terms of the A_∞-operations $m_i^A\colon A^{\otimes i} \longrightarrow A$ and $m_i^B\colon B^{\otimes i} \longrightarrow B$ contain a meaningless infinite summation when $f_0 \neq 0$ and one is working, e. g., over a field of coefficients (unless $m_i^B = 0$ for $i \gg 0$).

The explanation is that while the maps m_i^A are interpreted as the components of a coderivation of the tensor coalgebra cogenerated by the graded vector space A, the maps f_i are the components of a morphism between such tensor coalgebras. And while coderivations may not preserve coaugmentations of conilpotent coalgebras over fields, coalgebra morphisms always do.

0.11. – The latter problem can be solved by having f_0 divisible by the maximal ideal $\mathfrak{m}$ of a complete local ring $\mathfrak{R}$ and the components of the graded $\mathfrak{R}$-modules A and B complete in the $\mathfrak{m}$-adic topology, to make the relevant infinite sums convergent. One also wants the components of one's A_∞-algebras over $\mathfrak{R}$ to be free (complete) $\mathfrak{R}$-modules, so that their completed tensor product over $\mathfrak{R}$ is an exact functor.

This is a good definition of the category of curved A_∞-algebras to work with; but when dealing with A_∞-modules, it is useful to have an abelian category to which their components may belong. And the category of (infinitely generated) $\mathfrak{m}$-adically complete $\mathfrak{R}$-modules is *not* an abelian already for $\mathfrak{R} = k[[\epsilon]]$. The natural abelian category into which complete $\mathfrak{R}$-modules are embedded is that of $\mathfrak{R}$*-contramodules*.

In particular, when $\mathfrak{R} = \mathbb{Z}_l$ is the ring of l-adic integers, the abelian category of $\mathfrak{R}$-contramodules is that of the *weakly l-complete abelian groups* of Jannsen [**24**], known also as the *Ext-p-complete abelian groups* of Bousfield-Kan [**4**] (where $p = l$). Contramodules over $k[[\epsilon]]$ are very similar [**41**, Remarks A.1.1 and A.3]. Another name for contramodules over the adic completion of a Noetherian ring by an ideal is the *cohomologically complete modules* of Yekutieli et al. [**38, 39, 51**].

Contramodules over a topological ring $\mathfrak{R}$, particularly, a topological ring which does not contain any field (such as $\mathfrak{R} = \mathbb{Z}_l$), are defined as modules/algebras over a certain *monad on the category of sets* associated with $\mathfrak{R}$. Notice that a systematic study of a class of such monads, called the "algebraic" monads, was undertaken by Durov in [**8**]; however, the monads that appear in connection with contramodules are not algebraic, as they do not preserve filtered inductive limits of sets. We refer to the introduction to [**47**] for a detailed discussion and further references.

0.12. – Generally, contramodules are modules with infinite summation operations [**46**]. Among other things, they provide a way of having an abelian category of nontopological modules with some completeness properties over a coring or a topological ring. Defined originally by Eilenberg and Moore [**12**] as natural counterparts of comodules over coalgebras over commutative rings, contramodules were studied and used in the present author's monograph [**41**] for the purposes of the semi-infinite cohomology theory and the comodule-contramodule correspondence.

In particular, $k[[\epsilon]]$-contramodules form a full subcategory of the category of $k[[\epsilon]]$-modules (and even a full subcategory of the category of $k[\epsilon]$-modules). This subcategory contains all the $k[[\epsilon]]$-modules M such that $M \simeq \varprojlim_n M/\epsilon^n M$, and also some other $k[[\epsilon]]$-modules (hence the "weakly complete" terminology). The natural map $\mathfrak{M} \longrightarrow \varprojlim_n \mathfrak{M}/\epsilon^n\mathfrak{M}$ is surjective for every $k[[\epsilon]]$-contramodule $\mathfrak{M}$, but it may not be injective [**41**, Section A.1.1 and Lemma A.2.3].

The more familiar *comodules*, on the other hand, are basically discrete or torsion modules. For a pro-Artinian topological ring, they are defined as the opposite category to that of Gabriel's *pseudo-compact modules* [**17**]. Notice that our $\mathfrak{R}$-comodules are *not* literally discrete modules over a topological ring $\mathfrak{R}$, although any choice of an injective hull of the irreducible discrete module over a pro-Artinian commutative local ring $\mathfrak{R}$ provides an equivalence between these two abelian categories (and there is even a *natural* equivalence when $\mathfrak{R}$ is a profinite-dimensional algebra over a field or a profinite ring). So the $k[[\epsilon]]$-comodules are just $k[[\epsilon]]$-modules with a locally nilpotent action of ϵ.

More specifically, comodules over a pro-Artinian topological ring $\mathfrak{R}$ are defined in this memoir as ind-objects in the abelian category opposite to the category of discrete $\mathfrak{R}$-modules of finite length. We refer to the book [**26**] for a background discussion of ind-objects. Let us mention the similarity between this definition of ours and the notion of ind-coherent sheaves on ind-schemes and ind-inf-schemes, studied by Gaitsgory and Rozenblyum [**18, 19**]. One difference between the two approaches is that the ind-coherent sheaves in the sense of [**18, 19**] are *complexes* of sheaves; so they form a DG-category or a stable $(\infty, 1)$-category, i.e., a refined version of a triangulated category. Our $\mathfrak{R}$-comodules, on the other hand, form an abelian category.

The conventional formalism of tensor operations (i.e., the tensor product and Hom) on modules or bimodules over rings can be extended to comodules and contramodules over noncocommutative corings, where the natural operations are in much greater abundance and variety (there are five of them to be found in [**41, 42**]). For contramodules and comodules over a pro-Artinian commutative ring, we define the total of *seven* such operations in this memoir.

This allows to consider, in particular, curved A_∞-modules over $\mathfrak{R}$-free $\mathfrak{R}$-complete curved A_∞-algebras with, alternatively, either $\mathfrak{R}$-contramodule ("weakly complete") or $\mathfrak{R}$-comodule ("torsion," "discrete") coefficients. By another instance of the derived comodule-contramodule correspondence, the corresponding two homotopy categories are naturally equivalent.

0.13. – Yet another reason to work with complete modules or contramodules rather than just conventional modules over a local ring is the need to use Nakayama's lemma as the basic technical tool. The point is, the conventional version of Nakayama's lemma for modules over local rings only holds for finitely generated modules. Of course, one does not want to restrict oneself to finite-dimensional vector spaces or finitely generated modules over the coefficient ring when doing the homological algebra of A_∞-algebras and A_∞-modules.

So we want to have an abelian category of modules with infinite direct sums and products where Nakayama's lemma holds. Notice that Nakayama's lemma holds for infinitely generated modules over an Artinian local ring. As a natural generalization of this obvious observation, the appropriate version of Nakayama's lemma for infinitely generated contramodules over a topological ring with a topologically nilpotent maximal ideal was obtained in [**41**, Section A.2 and Remark A.3].

For comodules over a pro-Artinian local ring, we use the dual version of Nakayama's lemma, which is clearly true.

0.14. – We call curved DG-algebras in the tensor category of free $\mathfrak{R}$-contramodules with the curvature element divisible by the maximal ideal $\mathfrak{m} \subset \mathfrak{R}$ *weakly curved DG-algebras*, or *wcDG-algebras* over $\mathfrak{R}$. Morphisms of wcDG-algebras are CDG-algebra morphisms with the change-of-connection elements divisible by $\mathfrak{m}$.

CDG-modules over a wcDG-algebra are referred to as wcDG-modules. The similar terminology is used for A_∞-algebras: a *weakly curved* A_∞*-algebra*, or a *wc* A_∞*-algebra*, is a curved A_∞-algebra in the tensor category of free $\mathfrak{R}$-contramodules with the curvature element divisible by $\mathfrak{m}$.

So we can summarize much of the preceding discussion by saying that *theories* (i.e., derived categories and derived functors) *of the first kind make sense in the weakly curved, but not in the strongly curved case.*

0.15. – Let us explain how we define the equivalence relation on wcDG-modules and wc A_∞-modules. First assume that the underlying graded $\mathfrak{R}$-module of our weakly curved module $\mathfrak{M}$ over $\mathfrak{A}$ is a free graded $\mathfrak{R}$-contramodule.

In this case we simply apply the functor of reduction modulo $\mathfrak{m}$ to obtain an uncurved DG- or A_∞-module $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ over an uncurved algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$. The weakly curved module $\mathfrak{M}$ is viewed as a trivial object of our triangulated category of modules if the complex $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ is acyclic. In particular, when the curvature element of $\mathfrak{A}$ in fact vanishes, this condition means that the complex of free $\mathfrak{R}$-contramodules $\mathfrak{M}$ should be contractible (rather than just acyclic).

So the triangulated category of wcDG- or wc A_∞-modules that we construct is not actually their derived category of the first kind, but rather a mixed, or *semiderived category* [**41**]. It behaves as the derived category of the first kind in the direction of $\mathfrak{A}$ relative to $\mathfrak{R}$, and the derived category of the second kind, or more precisely the contraderived category, along the variables from $\mathfrak{R}$.

For this reason we call the weakly curved modules $\mathfrak{M}$ such that $\mathfrak{M}/\mathfrak{m}\mathfrak{M}$ is an acyclic complex of vector spaces *semiacyclic.* The prefix "semi" here means roughly "halfway between acyclic and contraacyclic" or even "halfway between acyclic and contractible"; so it should be thought of as a condition *stronger* than the acyclicity.

As to wcDG- or wc A_∞-modules $\mathfrak{N}$ over $\mathfrak{A}$ whose underlying graded $\mathfrak{R}$-modules are $\mathfrak{R}$-contramodules that are not necessarily free, we replace them with $\mathfrak{R}$-free wc $\mathfrak{A}$-modules $\mathfrak{M}$ isomorphic to $\mathfrak{N}$ in the contraderived category of weakly curved $\mathfrak{A}$-modules before reducing modulo $\mathfrak{m}$ to check for semiacyclicity. So our semiderived category of arbitrary $\mathfrak{R}$-contramodule wcDG- or wc A_∞-modules over $\mathfrak{A}$ is the quotient category of their contraderived category by the kernel of the derived reduction functor. To construct such an $\mathfrak{R}$-free resolution $\mathfrak{M}$ of a given weakly curved $\mathfrak{A}$-module $\mathfrak{N}$, it suffices to find an $\mathfrak{R}$-free left resolution for $\mathfrak{N}$ in the abelian category of $\mathfrak{R}$-contramodule weakly curved $\mathfrak{A}$-modules and totalize it by taking infinite products along the diagonals.

The definition of the equivalence relation on wcDG-modules or wc A_∞-modules over $\mathfrak{A}$ whose underlying graded $\mathfrak{R}$-modules are cofree $\mathfrak{R}$-comodules is similar, except that functor $\mathcal{M} \longmapsto {}_{\mathfrak{m}}\mathcal{M}$ of passage to the maximal submodule annihilated by $\mathfrak{m}$ is used to obtain a complex of vector spaces from a curved $\mathfrak{A}$-module in this case. And for arbitrary $\mathfrak{R}$-comodule weakly curved $\mathfrak{A}$-modules, the semiderived category is constructed as the quotient category of the coderived category of such modules by the kernel of the derived $\mathfrak{m}$-annihilated submodule functor.

0.16. – In particular, when the categories of $\mathfrak{R}$-contramodules and $\mathfrak{R}$-comodules have finite homological dimensions (e. g., $\mathfrak{R}$ is a regular complete Noetherian local ring), any acyclic complex of free $\mathfrak{R}$-contramodules or cofree $\mathfrak{R}$-modules is contractible. In this case, our semiderived category of wcDG- or wc A_∞-modules can be viewed as a true derived category of the first kind and called simply the *derived category.*

On the other hand, it is instructive to consider the case of the ring of dual numbers $R = k[\epsilon]/\epsilon^2$. In this case, there is no difference between R-contramodules and R-comodules, which are both just R-modules; and accordingly no difference between R-contramodule and R-comodule curved A-modules.

Still, their semiderived categories are different, in the sense that the two equivalence relations on weakly curved A-modules (in other words, the two classes of semiacyclic curved modules) are different in the case of weakly curved modules that are not (co)free over R. Indeed, they are different already for $A = R$, as the classes of coacyclic and contraacyclic complexes of R-modules are different [**42**, Examples 3.3].

The two semiderived categories of weakly curved A-modules are equivalent (as they are for any weakly curved algebra $\mathfrak{A}$ over any pro-Artinian local ring $\mathfrak{R}$), but the equivalence is a nontrivial construction when applied to modules that are not R-(co)free.

0.17. – The most striking aspect of the theory of (semi)derived categories of weakly curved modules developed in this memoir is just how nontrivial these are. On the one hand, there is a general tendency of the curvature to trivialize the categories of modules, well-known to the specialists now (see, e. g., [**29**]).

One reason for this is that apparently no curved modules over a given curved algebra can be pointed out a priori that would not be known to vanish in the homotopy category already. In particular, a curved algebra has *no* natural structure of a curved module over itself. Some explicit constructions of CDG-modules are used in the proofs of the general theorems about them in [**42, 37, 10**], but these always produce contractible CDG-modules. There are lots of examples of nontrivial CDG-modules, but these are CDG-modules over CDG-algebras of some special types (CDG-bimodules [**37**], Koszul CDG-algebras [**40**], change-of-connection transformations of DG-algebras, etc.).

In particular, an example from [**29**] shows that our (semi)derived category of weakly curved modules over a wcDG- or wc A_∞-algebra $\mathfrak{A}$ may vanish entirely even when the DG-algebra $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ has a nonzero cohomology algebra. This can already happen over $\mathfrak{R} = k[[\epsilon]]$, or indeed, over $\mathfrak{R} = k[\epsilon]/\epsilon^2$.

Specifically, let $\mathfrak{A} = \mathfrak{R}[x, x^{-1}]$ be the graded algebra over $\mathfrak{R}$ generated by an element x of degree 2 and its inverse element x^{-1}, with the only relation saying that these two elements should be inverse to each other, endowed with the zero differential, vanishing higher operations $m_i = 0$ for $i \geqslant 3$, and the curvature element $h = \epsilon x$. Then the homotopy categories of $\mathfrak{R}$-free and $\mathfrak{R}$-cofree wcDG-modules over $\mathfrak{A}$ already vanish, as consequently do the (semi)derived categories of wcDG- and wc A_∞-modules over $\mathfrak{A}$ (see Example 5.3.6).

For a similar wcDG-algebra $\mathfrak{A}' = \mathfrak{R}[x]$ with $\deg x = 2$, $d = 0$, and $h = \epsilon x$, one obtains a nonvanishing (semi)derived category of wcDG- or wc A_∞-modules in which all the $\mathfrak{R}$-contramodules of morphisms are annihilated by ϵ (see Example 6.6.1).

0.18. – On the other hand, the obvious expectation that the $\mathfrak{R}$-(contra)modules of morphisms in the triangulated category of weakly curved $\mathfrak{A}$-modules are always torsion modules is most emphatically *not true*. The reason for the obvious expectation is, of course, that the homotopy category of curved A_∞-modules is trivial over a field.

The explanation is that one cannot quite localize contramodules. The functor assigning to an $\mathfrak{R}$-contramodule the tensor product of its underlying $\mathfrak{R}$-module with the field of quotients of $\mathfrak{R}$ does *not* preserve either the tensor product or the internal Hom of contramodules; nor, indeed, does it preserve even infinite direct sums.

0.19. – Our computations of the $\mathfrak{R}$-contramodules Hom in certain (semi)derived categories of wcDG- and wc A_∞-modules are based on the Koszul duality theorems generalizing those in [**42**]. The semiderived category of wcDG-modules over a wcDG-algebra $\mathfrak{A}$ is equivalent to the coderived category of CDG-comodules and the contraderived category of CDG-contramodules over the CDG-coalgebra $\mathfrak{C} = \mathrm{Bar}(\mathfrak{A})$ obtained by applying the bar construction to $\mathfrak{A}$. Similarly, the co/contraderived category of CDG-co/contramodules over an $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$ that is conilpotent modulo $\mathfrak{m}$ is equivalent to the semiderived category of wcDG-modules over the cobar construction $\mathrm{Cob}(\mathfrak{C})$.

In particular, it follows that the semiderived category of wcDG-modules over a wcDG-algebra $\mathfrak{A}$ is equivalent to the semiderived category of wc A_∞-modules over $\mathfrak{A}$ considered as a wc A_∞-algebra, so our lumping together of the wcDG- and wc A_∞-modules in the preceding discussion is justified. On the other hand, the semiderived category of wc A_∞-modules over a wc A_∞-algebra $\mathfrak{A}$ is equivalent to the semiderived category of CDG-modules over the enveloping wcDG-algebra of $\mathfrak{A}$.

0.20. – To obtain a specific example of a nontrivial Hom computation in a (semi)derived category of wcDG- or wc A_∞-modules, one can start with an ungraded $\mathfrak{R}$-free coalgebra $\mathfrak{C}$ considered as a CDG-coalgebra concentrated in degree 0 with a zero differential and a zero curvature function. Then the co/contraderived category of, say, $\mathfrak{R}$-free CDG-co/contramodules over $\mathfrak{C}$ is just the co/contraderived category of the exact category of $\mathfrak{R}$-free $\mathfrak{C}$-co/contramodules. In particular, the exact category of $\mathfrak{R}$-free comodules over $\mathfrak{C}$ embeds into its coderived category (and similarly for contramodules).

So considering $\mathfrak{C}$ as a comodule over itself we obtain an example of an object in the coderived category whose endomorphism ring is a nonvanishing *free* $\mathfrak{R}$-contramodule. On the other hand, whenever the k-coalgebra $\mathfrak{C}/\mathfrak{m}\mathfrak{C}$ is conilpotent, by the Koszul duality theorems mentioned above the coderived category of $\mathfrak{R}$-free comodules over $\mathfrak{C}$ is equivalent to the semiderived category of wcDG- or wc A_∞-modules over $\mathrm{Cob}(\mathfrak{C})$. It remains to pick an $\mathfrak{R}$-free coalgebra $\mathfrak{C}$ that is conilpotent modulo $\mathfrak{m}$ but has no coaugmentation over $\mathfrak{R}$ in order produce an example of a semiderived category of wcDG- or wc A_∞-modules with a nonzero $\mathfrak{R}$-free Hom contramodule.

In the simplest case of the coalgebra dual to the $\mathfrak{R}$-free algebra of finite rank $\mathfrak{R}[y]/(y^2 = \epsilon)$ (with $\mathfrak{R} = k[[\epsilon]]$ or $k[\epsilon]/\epsilon^2$, as above) we obtain the wcDG-algebra $\mathfrak{A} = \mathrm{Cob}(\mathfrak{C})$ that is freely generated over $\mathfrak{R}$ by an element x of degree 1, with the zero differential and the curvature element $h = \epsilon x^2$. The cobar construction assigns to the cofree comodule $\mathfrak{C}$ over $\mathfrak{C}$ a wcDG-module $\mathfrak{M}$ over $\mathfrak{A}$ with an underlying graded $\mathfrak{A}$-module freely generated by two elements. The algebra of endomorphisms of $\mathfrak{M}$ in the (semi)derived category of wcDG- or wc A_∞-modules over $\mathfrak{A}$ is isomorphic to $\mathfrak{R}[y]/(y^2 = \epsilon)$, so it is a free $\mathfrak{R}$-contramodule of rank two (see Example 6.6.2).

0.21. – One of our most important results in this memoir is that the semiderived category wcDG- or wc A_∞-modules over a wcDG- or wc A_∞-algebra $\mathfrak{A}$ over $\mathfrak{R}$ is compactly generated. So are the coderived category of CDG-comodules and the contraderived category of CDG-contramodules over an $\mathfrak{R}$-free CDG-coalgebra $\mathfrak{C}$.

In the case of wcDG- or wc A_∞-modules we present a *single*, if not quite explicit, compact generator. In order to construct this generator, one has to consider $\mathfrak{R}$-comodule coefficients. In the semiderived category of $\mathfrak{R}$-comodule wcDG- or wc A_∞-modules over $\mathfrak{A}$, the weakly curved module $\mathfrak{A}/\mathfrak{m}\mathfrak{A}$ is the desired compactly generating object. As we have already mentioned, the semiderived categories of $\mathfrak{R}$-contramodule and

$\mathfrak{R}$-comodule weakly curved modules over $\mathfrak{A}$ are equivalent, but the equivalence is a somewhat complicated construction.

In the case of the coderived category of CDG-comodules over $\mathfrak{C}$, we also consider $\mathfrak{R}$-comodule coefficients, and have CDG-comodules whose underlying graded $\mathfrak{R}$-comodules have finite length form a triangulated subcategory of compact generators. For the contraderived category of CDG-contramodules over $\mathfrak{C}$, there is, once again, no explicit construction: one just has to identify this category with the coderived category of CDG-comodules.

While mainly interested in the curved modules and co/contramodules in the tensor category of $\mathfrak{R}$-contramodules, it is chiefly for the purposes of these constructions of compact generators that we pay so much attention to the $\mathfrak{R}$-comodule coefficients and the $\mathfrak{R}$-comodule-contramodule correspondence in our exposition.

0.22. – Mixing contramodules with comodules is a tricky business, though. We have already discussed $\mathfrak{R}$-contramodule weakly curved $\mathfrak{A}$-modules and $\mathfrak{R}$-comodule weakly curved $\mathfrak{A}$-module in this introduction; and now we have just mentioned $\mathfrak{R}$-comodule curved $\mathfrak{C}$-comodules. So let us use the occasion to *warn* the reader that, apparently, it makes little sense to consider arbitrary $\mathfrak{R}$-contramodule $\mathfrak{C}$-comodules or $\mathfrak{R}$-comodule $\mathfrak{C}$-contramodules, as such categories of graded modules are not even abelian, the relevant functors on the categories of $\mathfrak{R}$-contramodules and $\mathfrak{R}$-comodules not having the required exactness properties.

The (exotic derived) categories of $\mathfrak{R}$-free curved $\mathfrak{C}$-comodules and $\mathfrak{R}$-cofree curved $\mathfrak{C}$-contramodules make perfect sense and are well-behaved; and so are the categories of arbitrary $\mathfrak{R}$-contramodule (or just $\mathfrak{R}$-free) curved $\mathfrak{C}$-contramodules and arbitrary $\mathfrak{R}$-comodule (or just $\mathfrak{R}$-cofree) curved $\mathfrak{C}$-comodules. The (exotic derived) categories of $\mathfrak{R}$-contramodule or $\mathfrak{R}$-comodule (weakly) curved $\mathfrak{A}$-modules are also well-behaved, as are the categories of $\mathfrak{R}$-free or $\mathfrak{R}$-cofree (weakly) curved $\mathfrak{A}$-modules. But arbitrary (other than just $\mathfrak{R}$-free or $\mathfrak{R}$-cofree) $\mathfrak{R}$-contramodule $\mathfrak{C}$-comodules or $\mathfrak{R}$-comodule $\mathfrak{C}$-contramodules are generally problematic.

0.23. – To end, let us briefly discuss the motivation and possible applications. We are not in the position to suggest here any specific ways in which the techniques we are developing could be applied in Fukaya's Lagrangian Floer theory. In fact, the Novikov ring, which is the coefficient ring of the Floer-Fukaya theory, is *not* pro-Artinian (*nor* is it a topological local ring in our definition); so our results do not seem to be at present directly applicable.

Thus we restrict ourselves to stating that curved A_∞-algebras *do* seem to appear in the Floer-Fukaya business, and that their curvature (and change-of-connection) elements *do* seem to be, by the definition, divisible by appropriate maximal ideal(s) of the coefficient ring(s) [**15, 16, 6**]. Our study *does* imply that, generally speaking, quite nontrivial derived categories of modules can be associated with curved algebras of this kind. Working over a complete local ring, rather than over a field, is the price one has to pay for being able to obtain these derived categories of modules.

Furthermore, the semiderived categories of weakly curved DG- and A_∞-modules have all the usual properties of the derived categories of DG- and A_∞-modules over algebras over fields. The only caveat is that the nontriviality is *not guaranteed*: the triangulated categories of weakly curved modules may sometimes vanish when, on the basis of the experience with uncurved modules over algebras over fields, one would not expect them to (as it was noticed in [**29**]).

0.24. – Another possible application has to do with the deformation theory of DG-algebras. As pointed out in [**29**], if one presumes that the deformations of DG-algebras should be controlled by their Hochschild cohomology complexes, one discovers that deformations in the class of CDG-algebras are to be considered on par with the conventional DG-algebra deformations.

A curved infinitesimal or formal deformation of a DG-algebra A over a field k is a wcDG-algebra $\mathfrak{A}$ over the ring $R = k[\epsilon]/\epsilon^2$ or $\mathfrak{R} = k[[\epsilon]]$, respectively. The problem of constructing deformations of the derived categories of DG-modules corresponding to curved deformations of DG-algebras was discussed in [**29**] (cf. the recent paper [**7**]).

Without delving into the implications of the deformation theory viewpoint, let us simply state that what seems to be a reasonable definition of the conventional derived category of wcDG-modules in the case of a pro-Artinian topological local ring $\mathfrak{R}$ of finite homological dimension is developed in this memoir. So, at least, the case of a formal deformation may be (in some way) covered by our theory.

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