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# DISCRETE GEOMETRY AND ISOTROPIC SURFACES


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# DISCRETE GEOMETRY AND ISOTROPIC SURFACES

**François Jauberteau, Yann Rollin, Samuel Tapie**

***Abstract.*** – We consider smooth isotropic immersions from the 2-dimensional torus into $\mathbb{R}^{2n}$, for $n \geqslant 2$. When $n = 2$ the image of such map is an immersed Lagrangian torus of $\mathbb{R}^4$. We prove that such isotropic immersions can be approximated by arbitrarily $\mathcal{C}^0$-close piecewise linear isotropic maps. If $n \geqslant 3$ the piecewise linear isotropic maps can be chosen so that they are piecewise linear isotropic immersions as well.

The proofs are obtained using analogies with an infinite dimensional moment map geometry due to Donaldson. As a byproduct of these considerations, we introduce a numerical flow in finite dimension, whose limits provide, from an experimental perspective, many examples of piecewise linear Lagrangian tori in $\mathbb{R}^4$. The DMMF program, which is freely available, is based on the Euler method and shows the evolution equation of discrete surfaces in real time, as a movie.

***Résumé*** **(Géométrie discrète et surfaces isotropes).** – Nous considérons des immersions lisses et isotropes du tore de dimension 2 vers $\mathbb{R}^{2n}$, pour $n \geqslant 2$. Quand $n = 2$ l'image d'une telle application est un tore lagrangien immergé de $\mathbb{R}^4$. Nous démontrons que de telles immersions isotropes peuvent être approximées au sens $\mathcal{C}^0$, par des applications linéaires par morceaux et isotropes arbitrairement proches. Si $n \geqslant 3$, il est possible des choisir des applications linéaires par morceaux qui sont de plus des immersions.

Les démonstrations reposent sur des analogies avec une géométrie et une application moment en dimension infinie introduites par Donaldson. Nous en déduisons un flot en dimension finie, dont les limites, du point de vue expérimental, produisent de nombreux exemples de tores lagrangiens linéraires par morceaux de $\mathbb{R}^4$. Le programme libre DMMF, basé sur la méthode d'Euler, montre l'équation d'évolution sous forme de film.

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# CHAPTER 1

# INTRODUCTION

## 1.1. Original motivations and background

*Lagrangian submanifolds* are natural objects, arising in the context of *Hamiltonian mechanics* and *dynamical systems*. Their prominent role in symplectic topology and geometry should not come as a surprise. In spite of tremendous efforts, the classification of Lagrangian submanifolds, up to *Hamiltonian isotopy*, is generally an open problem: for instance, Lagrangian tori of the Euclidean symplectic space $\mathbb{R}^4$ are not classified up to Hamiltionian isotopy. Lagrangian submanifolds are also key objects of various gauge theories. For example, the *Lagrangian Floer theory* is defined by counting pseudoholomorphic disks with boundary contained in some prescribed Lagrangian submanifolds. Many examples of *smooth* Lagrangian submanifolds are known. They are easy to construct and to deform. In a nutshell, Lagrangian submanifolds are typical, rather *flexible* objects, from symplectic topology.

An elementary construction of Lagrangian submanifold is provided by considering the 0-section of the cotangent bundle of a smooth manifold $T^*L$, endowed with its natural symplectic structure $\omega = d\lambda$, where $\lambda$ is the *Liouville form*. More generally, it is well known that any section of $T^*L$ given by a closed 1-form is a Lagrangian submanifold. Furthermore, such Lagrangian submanifolds are Hamiltonian isotopic to the 0-section if, and only if, the corresponding 1-form is exact. These examples provide a large class of Lagrangian submanifolds which admit as many Hamiltonian deformations as smooth function on $L$ modulo constants.

By the *Lagrangian neighborhood theorem*, every Lagrangian submanifold $L$ of a symplectic manifold admits a neighborhood symplectomorphic to a neighborhood of the 0-section of $T^*L$. It follows that the local Hamiltonian deformations discussed above (in the case of $T^*L$) also provide deformations for Lagrangian submanifolds of *any* symplectic manifold.

The geometric notion of *stationary Lagrangians submanifolds* was introduced by Oh [**8, 9**] in order to seek canonical representatives, in a given isotopy class of Lagrangian submanifolds. Stationary Lagrangian submanifolds can be though of as analogs of *minimal submanifolds* in the framework of symplectic geometry. Stationary Lagrangians are expected to be canonical in some sense, and Oh conjectured for instance that Clifford tori of $\mathbb{CP}^2$ should minimize the volume in their Hamiltonian isotopy class.

As in the case of minimal surfaces, one can define various modified versions of the *mean curvature flow*, which are expected to converge toward stationary Lagrangians submanifolds in a given isotopy class. In an attempt to implement numerical versions of these flows [**5**], we ended up facing theoretical problems of a *discrete geometric nature*. Indeed, from a numerical point of view, surfaces are usually understood as some type of *mesh* and their mathematical counterpart is *discrete geometry* and sometimes *piecewise linear* geometry. Two obstacles arose in order to provide a sound numerical simulation of geometric flows for Lagrangian submanifolds, namely:

1. To the best of our knowledge, discrete Lagrangian surfaces of $\mathbb{R}^4$ and more generally discrete isotropic surfaces of $\mathbb{R}^{2n}$ are poorly understood, in fact hardly studied. We had no available examples of discrete Lagrangian tori in $\mathbb{R}^4$ in our toolbox, save some discrete analogs of product or Chekanov tori (cf. Section 3.5). Furthermore, we had no deformation theory that we could rely upon contrarily to the smooth case. Implementing a geometric evolution equation for discrete Lagrangian surfaces, with so few examples to start the flow was not an enticing project.
2. As far as a program is based on a numerical implementation, using floating point numbers, it is not natural to check if a symplectic form vanishes exactly along a plane. It only makes sense to test if the symplectic density is rather small, which means that we have an approximate solution of our problem. From an experimental point of view, we dread our numerical flow would exhibit some spurious drift of the symplectic density. We feared such instabilities may jeopardize our numerical simulations for flowing Lagrangian submanifolds.

These issues led us to consider an auxiliary flow. Ideally, the auxiliary flow should attract *any* discrete surface toward Lagrangian discrete surfaces. The utility of the auxiliary flow would be 2-fold: its limits would provide examples of Lagrangian discrete surfaces for our experiments. It may also be used to prevent instabilities of evolution equation along the moduli space of discrete Lagrangian surfaces.

These questions are part of a larger ongoing project. They have not been fully investigated yet but stirred many questions of a discrete differential geometric nature, in the context of symplectic geometry. This paper delivers a few answers to some of the simplest questions arising, as a spin-off to our initial motivations.

## 1.2. Statement of results

We consider smooth maps $\ell : \Sigma \to \mathbb{R}^{2n}$, where $\Sigma$ is a surface and $n \geqslant 2$. The Euclidean space $\mathbb{R}^{2n}$ is endowed with its standard symplectic form $\omega$. A map $\ell$ is said to be *isotropic* if $\ell^*\omega = 0$. Lagrangian tori of $\mathbb{R}^4$ are the submanifolds obtained as the image of $\Sigma$ by $\ell$, in the particular case where $2n = 4$, $\Sigma$ is diffeomorphic to a torus and $\ell$ is an isotropic embedding.

In this paper, we construct approximations of smooth isotropic immersions of the torus in $\mathbb{R}^{2n}$ by *piecewise linear isotropic maps*. The idea is to consider a discretization of the torus by a square grid and approximate the smooth map by a quadrangular mesh. This mesh is almost isotropic, in a suitable sense. A perturbative argument shows that there exists a nearby isotropic quadrangular mesh, which is used to build a piecewise linear map. We provide a more precise statement of the above claims in the rest of the introduction.

**1.2.1. Piecewise linear isotropic maps. –** We recall some usual definitions before stating one of our main results. A *triangulation* of $\mathbb{R}^2$ is a locally finite *simplicial complex* that covers $\mathbb{R}^2$ entirely. In this paper, points, line segments, triangles of triangulations are understood as geometrical Euclidean objects of the plane. Similarly, we shall consider triangulations of quotients of $\mathbb{R}^2$ by a lattice $\Gamma$, obtained by quotient of $\Gamma$-invariant triangulations of $\mathbb{R}^2$.

A *piecewise linear map* $f : \mathbb{R}^2 \to \mathbb{R}^m$ is a continuous map such that, for some triangulation of $\mathbb{R}^2$, the restriction of $f$ to any triangle is an affine map to $\mathbb{R}^m$.

We consider smooth isotropic immersions $\ell : \Sigma \to \mathbb{R}^{2n}$, where $\Sigma$ is diffeomorphic to a 2-dimensional torus and $n \geqslant 2$. The Euclidean metric $g$ of $\mathbb{R}^{2n}$ induces a conformal structure on $\Sigma$. The uniformization theorem implies that the conformal structure of $\Sigma$ actually comes from a quotient of $\mathbb{R}^2$, with its canonical conformal structure, by a lattice. Thus, we have a conformal covering map

$$p : \mathbb{R}^2 \to \Sigma,$$

with group of deck transformations $\Gamma$, a lattice of $\mathbb{R}^2$.

A triangulation (resp. quadrangulation) of $\Sigma$ is called an *Euclidean triangulation* (resp. *quadrangulation*) of $\Sigma$ if the boundary of every face lifts to an Euclidean triangle (resp.quadrilateral) of $\mathbb{R}^2$ via $p$.

Similarly, a function $f : \Sigma \to \mathbb{R}^m$ is a *piecewise linear map* if it lifts to a piecewise linear map $\mathbb{R}^2 \to \mathbb{R}^m$ via $p$. Given a piecewise linear map $\hat{\ell} : \Sigma \to \mathbb{R}^{2n}$, the pull-back of the symplectic form $\omega$ of $\mathbb{R}^{2n}$ makes sense on each triangle of the triangulation subordinate to $\hat{\ell}$. We say that $\hat{\ell}$ is a *isotropic piecewise linear map* if the pull back of $\omega$ vanishes along each face of the triangulation. A piecewise linear map which is locally injective is called a *piecewise linear immersion.*

The main result of this paper can be stated as follows:

THEOREM A. – *Let $\ell : \Sigma \to \mathbb{R}^{2n}$ be a smooth isotropic immersion, where $\Sigma$ is a surface diffeomorphic to a compact torus and $n \geqslant 2$. Then, for every $\varepsilon > 0$, there exists a piecewise linear isotropic map $\hat{\ell} : \Sigma \to \mathbb{R}^{2n}$ such that for every $x \in \Sigma$, we have*

$$\|\ell(x) - \hat{\ell}(x)\| \leqslant \varepsilon.$$

*Furthermore, if $n \geqslant 3$, we may assume that $\hat{\ell}$ is an immersion. If $n = 2$, we may assume that $\hat{\ell}$ is an immersion away from a finite union of embedded circles in $\Sigma$.*

Loosely stated, Theorem A says that every isotropic immersion $\ell$ of a torus into $\mathbb{R}^{2n}$ can be approximated by a piece linear map arbitrarily $\mathscr{C}^0$-close to $\ell$. If $n \geqslant 3$ the last statement of the theorem provides the following corollary:

COROLLARY B. – *Let $n$ be an integer such that $n \geqslant 3$. Let $\Sigma$ be a smoothly immersed surface in $\mathbb{R}^{2n}$, which is isotropic and diffeomorphic to a compact torus. Then, there exist piecewise linear immersed surfaces in $\mathbb{R}^{2n}$, which are isotropic, homeomorphic to a compact torus and arbitrarily close to $\Sigma$ with respect to the Hausdorff distance.*

REMARK 1.2.2. – Our technique does not allow to get much better results than a rather rough $\mathscr{C}^0$-closedness between $\ell$ and its approximation $\hat{\ell}$. The best evidence for this weakness is the existence of a certain *shear action* on the space of isotropic quadrangular meshes (cf. § 4.2). It would be most interesting to understand whether these limitations are inherent to the techniques we employed here, or if there are geometric obstructions to get better estimates.

**1.2.3. Isotropic quadrangular meshes. –** The main tool to prove Theorem A relies on *quadrangulations* of $\Sigma$ and *quadrangular meshes*. Quadrangulations of $\mathbb{R}^2$ are particular CW-complex decompositions of $\mathbb{R}^2$, where edges are line segments of $\mathbb{R}^2$ and the boundary of every face is an Euclidean quadrilateral. Nevertheless, the precise general definition of quadrangulations is unimportant for our purpose. Indeed, we shall only work with particular standard quadrangulations $\mathscr{Q}_N(\mathbb{R}^2)$ of $\mathbb{R}^2$, pictured as a regular grid with step size $N^{-1}$ tiled by Euclidean squares.

Particular Euclidean quadrangulations of $\mathscr{Q}_N(\Sigma)$, are defined at § 3.3. They are obtained as quotients of $\mathscr{Q}_N(\mathbb{R}^2)$ by certain lattices $\Gamma_N$ of $\mathbb{R}^2$. The associated moduli space of *quadrangular meshes* is by definition

$$\mathscr{M}_N = C^0(\mathscr{Q}_N(\Sigma)) \otimes \mathbb{R}^{2n}.$$

A mesh $\tau \in \mathscr{M}_N$ is an object that associates $\mathbb{R}^{2n}$-coordinates to every vertex of the quadrangulation $\mathscr{Q}_N(\Sigma)$.

We would like to say that any quadrilateral of $\mathbb{R}^{2n}$ contained in an isotropic plane is an *isotropic quadrilateral*. However, quadrilaterals are generally not contained in a 2-dimensional plane. The above attempt of definition can be extended via the Stokes theorem for every non flat quadrilateral: a quadrilateral of $\mathbb{R}^{2n}$ is called an *isotropic quadrilateral*, if the integral of the Liouville form $\lambda$ along the quadrilateral — that is four oriented line segments — vanishes (cf. § 4.1).

REMARK 1.2.4. – In particular, for any compact embedded oriented surface $S$ of $\mathbb{R}^{2n}$ with boundary given by an isotropic quadrilateral, we have $\int_S \omega = 0$ by Stokes theorem.

By extension, we say that a mesh $\tau \in \mathscr{M}_N$ is isotropic if the quadrilateral in $\mathbb{R}^{2n}$ associated to each face of $\mathscr{Q}_n(\Sigma)$ via $\tau$ is isotropic. The main strategy for proving Theorem A involves the following approximation result:

THEOREM C. – *Given an isotropic immersion $\ell : \Sigma \to \mathbb{R}^{2n}$, there exists a family of isotropic quadrangular meshes $\rho_N \in \mathscr{M}_N$ defined for every $N$ sufficiently large, with the following property: for every $\varepsilon > 0$, there exists $N_0 > 0$ such that for every $N \geqslant N_0$ and every vertex $v$ of $\mathcal{Q}_N(\Sigma)$, we have*

$$\|\rho_N(v) - \ell(v)\| \leqslant \varepsilon.$$

An isotropic quadrilateral of $\mathbb{R}^{2n}$ is always the base of an isotropic pyramid in $\mathbb{R}^{2n}$ (cf. § 7.1), which is easily found as the solution of a linear system. This remark allows to pass from an isotropic quadrangular mesh to an isotropic triangular mesh. Together with Theorem C this provides essentially the proof of Theorem A.

**1.2.5. Flow for quadrangular meshes. –** Our approach for proving Theorem C has been inspired to a large extent by the beautiful *moment map geometry* introduced by Donaldson [**4**]. We shall provide a careful presentation of this infinite dimensional geometry at § 2, and merely state a few facts in this introduction: the moduli space of maps

$$\mathscr{M} = \{f : \Sigma \to \mathbb{R}^{2n}\},$$

from a surface $\Sigma$ endowed with a volume form $\sigma$ into $\mathbb{R}^{2n}$ admits a natural formal Kähler structure, with a formal Hamiltonian action of $\mathrm{Ham}(\Sigma, \sigma)$. The moment map of the action is given by

$$\mu(f) = \frac{f^*\omega}{\sigma}.$$

Zeroes of the moment map are precisely isotropic maps. It is tempting to make an analogy with the Kempf-Ness theorem, which holds in the finite dimensional setting. We may conjecture that a map $f$ admits an isotropic representative in its *complexified orbit* provided some type of algebro-geometric hypothesis of stability. Furthermore, one can also define a *moment map flow*, which is naturally defined in the context of a Kähler manifold endowed with a Hamiltonian group action. Such flow is essentially the downward gradient of the function $\|\mu\|^2$ on the moduli space, which is expected, in favorable circumstances, to converge toward a zero of the moment map in a prescribed orbit.

REMARK 1.2.6. – We shall not state any significant results aside the description of this geometric framework. For instance, it is an open question whether the moment map flow exists for short time in this context, which is part of a broader ongoing program.

In an attempt to define a finite dimensional analog of this infinite dimensional moment map picture, we define a flow analogous to the moment map flow on the moduli space of meshes $\mathscr{M}_N$, called the discrete moment map flow. This flow is now just an ODE and its behavior can readily be explored from a numerical perspective, using the Euler method. We provide a computer program called DMMF, available on the homepage

`http://www.math.sciences.univ-nantes.fr/~rollin/index.php?page=flow`, which is a numerical simulation of the discrete moment map flow. From an experimental point of view, the flow seems to be converging quickly toward isotropic quadrangular meshes, for any initial quadrangular mesh (cf. § 8).

## 1.3. Open questions

Theorem A is a fundamental tool for the discrete geometry of isotropic tori, since it provides a vast class of examples of piecewise linear objects by approximation of the smooth ones. Here is a list a questions that arise immediately in this new territory of discrete symplectic geometry:

1. Is there a converse to Theorem A or Corollary B? Given a piecewise linear isotropic surface in $\mathbb{R}^{2n}$, is it possible to find a nearby smooth isotropic surface?
2. More generally, to what extent does the moduli space of piecewise linear Lagrangian submanifolds retain the properties of the moduli space of smooth Lagrangian submanifolds? In spite of groundbreaking progress in symplectic topology, the classification of Lagrangian submanifolds up to Hamiltonian isotopy remains open. It is known that there exists several types of Lagrangian tori in $\mathbb{R}^4$, which are not Hamiltonian isotopic: namely, product tori and Chekanov tori [**1**]. On the other hand, Luttinger found infinitely many obstructions in [**7**] to the existence of certain type of knotted Lagrangian tori in $\mathbb{R}^4$. In particular spin knots provide knotted tori in $\mathbb{R}^4$ which cannot by isotopic to Lagrangian tori according to Luttinger's theorem. This thread of ideas led to the conjecture that product and Checkanov tori are the only classes of Lagrangian tori in $\mathbb{R}^4$, up to Hamiltonian isotopy. Although the result was claimed before, the conjecture is still open for the time being [**2**]. However it was proved by Dimitroglou Rizell, Goodman and Ivrii that all embedded Lagrangian tori of $\mathbb{R}^4$ are isotopic trough Lagrangian isotopies [**3**]. Perhaps an interesting approach to tackle such conjecture, and more generally any questions involving some type of $h$-principle, would be to recast the question in the finite dimensional framework of piecewise linear Lagrangian tori of $\mathbb{R}^4$.
3. The moment map framework, in an infinite dimensional context, presented at § 2, has been a great endeavor for proving our main results and introducing a finite dimensional version of the moment map flow. However, only a faint shadow of the moment map geometry is recovered in the finite dimensional world. More precisely, there exists a finite dimensional analog $\mu^r_N$ of the moment map $\mu$ on $\mathscr{M}_N$. But it is not clear whether $\mu^r_N$ is actually a moment map and for which group action on $\mathscr{M}_N$. It would be most interesting to define a finite dimensional analog of the group $\mathrm{Ham}(\Sigma, \sigma)$, and try to make sense of the Kempf-Ness theorem in this setting.