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NON-COMPACT FORM OF THE ELEMENTARY DISCRETE INVARIANT

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ABSTRACT. — We determine the non-compact form of Vishik's elementary discrete invariant for quadrics. As an application, we obtain new restrictions on the possible values of the elementary discrete invariant by studying the action of Steenrod operations on the algebraic cycles defining the non-compact form.

RÉSUMÉ (*Forme non-compacte de l'invariant discret élémentaire*). — On détermine la forme non-compacte de l'invariant discret élémentaire de Vishik pour quadriques. Comme application, on obtient de nouvelles restrictions sur les valeurs possibles de l'invariant discret élémentaire en étudiant l'action des opérations de Steenrod sur les cycles algébriques définissant la forme non-compacte.

1. Introduction

Let X be a smooth projective quadric of dimension n over a field F associated with a non-degenerate F-quadratic form q. The *splitting pattern* of X is a discrete invariant that measures the possible Witt indices of q_E over all field extensions E/F (see [4] and [5]). The *motivic decomposition type* of X is a discrete invariant which measures in what pieces the Chow motive of X can be

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decomposed. Moreover, Alexander Vishik noticed in [6] that the study of the interaction between these two invariants provides further information about both of them.

For this reason, he introduced the *generic discrete invariant* of quadrics, a bigger discrete invariant containing the splitting pattern and the motivic decomposition type invariants as faces, see [7] and [9]. The Generic Discrete Invariant $GDI(X)$ is defined as follows. Let K/F be a splitting field extension of q. Let us denote $[n/2]$ as d. For any $i \in \{0,\dots,d\}$, we write G_i for the grassmannian of i-dimensional totally q-isotropic subspaces (in particular G_0 is the quadric X). Then $GDI(X)$ is the collection of the subalgebras of rational elements

$$\overline{\mathrm{Ch}}^*(G_i) := \mathrm{Image}\left(\mathrm{Ch}^*(G_i) \to \mathrm{Ch}^*(G_{iK})\right)$$

for $i \in \{0,\dots,d\}$, where Ch stands for the Chow ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients (an algebraic cycle already defined at the level of the base field F is called *rational*).

In his paper [10] dedicated to the Kaplansky's conjecture on the u-invariant of a field, A. Vishik used the *elementary discrete invariant* of quadrics, a handier invariant than the GDI as it only deals with some particular cycles in $\mathrm{Ch}^*(G_{iK})$. More precisely, for any $i \in \{0,\dots,d\}$, we denote by $\mathcal{F}(0,i)$ the partial orthogonal flag variety of q-isotropic lines contained in i-dimensional totally q-isotropic subspaces. One can consider the diagram

$$X \xleftarrow[\pi_{(\underline{0},i)}]{} \mathcal{F}(0,i) \xrightarrow[\pi_{(0,\underline{i})}]{} G_i,$$

given by the natural projections and, for $0 \leq j \leq d$, we set

$$Z^i_{n-i-j} := \pi_{(0,\underline{i})_*} \circ \pi^*_{(\underline{0},i)}(l_j) \in \mathrm{CH}^{n-i-j}\left(G_{iK}\right),$$

where CH stands for the Chow ring with $\mathbb{Z}$-coefficients and l_j is the class in $\mathrm{CH}_j(X_K)$ of a j-dimensional totally isotropic subspace of $\mathbb{P}\left((V_q)_K\right)$ (with V_q the F-vector space associated with q). We set $z^i_{n-i-j} := Z^i_{n-i-j} \pmod 2 \in \mathrm{Ch}^{n-i-j}(G_{iK})$, with Ch being the Chow ring with $\mathbb{Z}/2\mathbb{Z}$-coefficients. The cycles z^i_{n-i-j} are the elementary classes defining the elementary discrete invariant $EDI(X)$:

Definition 1.1. — The *elementary discrete invariant* $EDI(X)$ is the collection of subsets $EDI(X,i)$ consisting of those integers m such that z^i_m is rational.

Furthermore, for any $r \geq 1$, the Chow motive of X^r with $\mathbb{Z}/2\mathbb{Z}$-coefficients decomposes into a direct sum of shifts of the motive of some G_i, see [2, Corollary 91.8]. Therefore, knowing $GDI(X)$ is the equivalent to knowing

$$\overline{\mathrm{Ch}}^*(X^r) := \mathrm{Image}\left(\mathrm{Ch}^*(X^r) \to \mathrm{Ch}^*(X^r_K)\right)$$

for all $r \geq 1$. Hence, the collection of the latter subalgebras constitutes a *non-compact* (in the sense that one has to consider infinitely many objects) form of $GDI(X)$. For the same reason, there exists a non-compact form of $EDI(X)$ (with defining cycles living in $\mathrm{Ch}^*(X_K^r)$), which we determine in the this work: for any $i \in \{0, \dots, d\}$, let us denote by $\mathrm{sym} : \mathrm{CH}^*(X^{i+1}) \to \mathrm{CH}^*(X^{i+1})$ the homomorphism $\Sigma_{s \in S_{i+1}} s_*$, where $s : X^{i+1} \to X^{i+1}$ is the isomorphism associated with a permutation s. For $0 \leq j \leq d$, we set

$$\rho_{i,j} := \mathrm{sym}\left(\left(\times_{k=0}^{i-1} h^k\right) \times l_j\right) \in \mathrm{CH}^{n-j+i(i-1)/2}\left(X_K^{i+1}\right),$$

where $\times$ is the external product and h^k is the k-th power of the hyperplane section class $h \in \mathrm{CH}^1(X)$ (always rational). Note that $\rho_{0,j} = Z_{n-j}^0 = l_j$. The symmetric cycles $\rho_{i,j}$ (mod 2) are the classes defining the non-compact form of $EDI(X)$:

Theorem 1.2. — *Let $1 \leq i \leq d$ and $0 \leq j \leq d$. The cycle z_{n-i-j}^i is rational if and only if the cycle $\rho_{i,j}$ $(mod\, 2)$ is rational.*

Because of the stability of rational cycles under pull-backs of diagonal morphisms and the possibility of a refined use of Steenrod operations of cohomological type, studying the non-compact form provides new restrictions on the possible values of $EDI(X)$, as illustrated by Sections 4 and 5.

Moreover, Theorem 1.2 reduces certain questions about the rationality of algebraic cycles on orthogonal grassmannians to the sole level of quadrics. For example, it allows one to reformulate both Vishik's conjecture [8, Conjecture 3.11] and the conjecture [9, Conjecture 0.13] on the *dimensions of Bruno Kahn*.

In Section 2, we introduce some basic tools which are required in Section 3, where we prove Theorem 1.2, using mainly compositions of correspondences and Chern classes of vector bundles over orthogonal grassmannians.

2. Preliminaries

In this section, we continue to use the notation introduced in Section 1.

2.1. Rational cycles on powers of quadrics. — We refer to [2, § 68] for an introduction to cycles on powers of quadrics. For any $1 \leq i \leq d$ and $0 \leq j \leq i-1$, we set

$$\Delta_{i,j} := \mathrm{sym}\left(\left(\times_{k=0}^{i-1} h^k\right) \times l_j\right) + \sum_{m=i}^{d} \mathrm{sym}\left(\left(\times_{\substack{k=0\\k\neq j}}^{i-1} h^k\right) \times h^m \times l_m\right)$$

in $\mathrm{Ch}^{n-j+i(i-1)/2}(X_K^{i+1})$. If $n = 2d$, we choose an orientation l_d of the quadric.

Lemma 2.1. — *For any $1 \leq i \leq d$ and $0 \leq j \leq i-1$, the cycle $\Delta_{i,j}$ is rational.*

Proof. — We proceed by induction on i. In $\mathrm{Ch}^n(X_K^2)$, the cycle $\Delta_{1,0}$ or $\Delta_{1,0}+h^d\times h^d$, depending on whether $l_d^2=0$ or not, is the class of the diagonal. Therefore, the cycle $\Delta_{1,0}$ is rational. Let $\sigma\in S_{i+1}$ be a cyclic permutation (with $i\geq 2$). For $0\leq j\leq i-2$, the induction hypothesis step is provided by the identity

$$\Delta_{i,j}=\sum_{l=0}^{i}\sigma_*^l\left(\Delta_{i-1,j}\times h^{i-1}\right)\quad\text{in}\quad \mathrm{Ch}\left(X_K^{i+1}\right).$$

It just remains to show that the cycle $\Delta_{i,i-1}$ is rational to complete the proof. In $\mathrm{Ch}(X_K^{i+1})$, one has

$$\begin{aligned}\Delta_{i,i-1}&=\sum_{m=i-1}^{d}\mathrm{sym}\left(\left(\times_{k=0}^{i-2}h^k\right)\times l_m\times h^m\right)\\&=\sum_{m=0}^{d}\mathrm{sym}\left(\left(\times_{k=0}^{i-2}h^k\right)\times l_m\times h^m\right)\end{aligned}$$

and the latter sum can be rewritten as

$$\sum_{s\in A_{i+1}}s_*\left(\left(\times_{k=0}^{i-2}h^k\right)\times\Delta_{1,0}\right).$$

Thus, the cycle $\Delta_{i,i-1}$ is rational. □

2.2. Correspondences. — We refer to [2, § 62] for an introduction to Chow-correspondences.

For any $1\leq i\leq d$, we denote by θ_i the class of the subvariety

$$\{(y,x_1,\dots,x_{i+1})\mid x_1,\dots x_{i+1}\in y\}\subset G_i\times X^{i+1}$$

in $\mathrm{CH}(G_i\times X^{i+1})$ and we view the cycle θ_i as a correspondence $G_i\rightsquigarrow X^{i+1}$.

We set

$$\eta_i:=\prod_{k=1}^{i}\left(\mathrm{Id}_{G_i}\times p_{X_k^i}\right)^*([\mathcal{F}(i,0)])\in\mathrm{CH}(G_i\times X^i),\tag{1}$$

with $p_{X_k^i}$ the projection from X^i to the k-th coordinate. For any integer $i\leq s\leq d$, we write

$$W_{s-i}^i:=\pi_{(0,\underline{i})_*}\circ\pi_{(\underline{0},i)}^*(h^s)\in\mathrm{CH}^{s-i}(G_i),$$

and $w_{s-i}^i:=W_{s-i}^i \pmod 2\in\mathrm{Ch}^{s-i}(G_i)$. Since the variety X_K is cellular, the cycle $[\mathcal{F}(i,0)]$ decomposes as

$$[\mathcal{F}(i,0)]=\sum_{s=0}^{d}z_{n-i-s}^i\times h^s+\sum_{s=i}^{d}w_{s-i}^i\times l_s\ \text{in}\ \mathrm{Ch}(G_{iK}\times X_K),\tag{2}$$

where l_d has to be replaced by the other class l'_d of maximal totally isotropic subspaces if $n = 2d$ and l_d^2 is not zero, i.e., if four divides n (see [2, Theorem 66.2]).

The two following lemmas, where we write p with underlined target for projections, can be proven the same way [3, Lemmas 3.2 and 3.10] have been proven but with Z^i_{n-i-j} (resp. z^i_{n-i-j}) instead of Z^i_{n-i} (resp. z^i_{n-i}).

Lemma 2.2. — *For any $1 \leq i \leq d$, $0 \leq j \leq d$ and $x \in CH(X_K)$, one has*

$$\left((\theta_i)_*(Z^i_{n-i-j})\right)_*(x) = p_{G_i \times \underline{X^i}\,*}\left(p^*_{\underline{G_i} \times X^i}\left(\pi_{(0,\underline{i})\,*} \circ \pi^*_{(\underline{0},i)}(x) \cdot Z^i_{n-i-j}\right) \cdot \eta_i\right),$$

where the cycle $(\theta_i)_*(Z^i_{n-i-j})$ is viewed as a correspondence $X_K \rightsquigarrow X^i_K$.

For any $1 \leq i \leq d$, we write $\mathcal{F}(i-1,i)$ for the partial orthogonal flag variety of $(i-1)$-dimensional totally isotropic subspaces contained in i-dimensional totally isotropic subspaces and we consider the diagram

$$G_{i-1} \xleftarrow[\pi_{(\underline{i-1},i)}]{} \mathcal{F}(i-1,i) \xrightarrow[\pi_{(i-1,\underline{i})}]{} G_i,$$

given by the natural projections.

Lemma 2.3. — *For any $2 \leq i \leq d$, $0 \leq j \leq d$ and $i \leq m \leq d$, the cycle*

$$p_{G_i \times \underline{X^i}\,*}\left(w^i_{m-i} \cdot z^i_{n-i-j} \cdot \eta_i\right) \in Ch(X^i_K),$$

where we write η_i for η_i $(mod\, 2)$, and this can be rewritten as

$$\sum_{s=0}^{m} \sum_{k=max(i-s,0)}^{min(m-s,i)} p_{G_{i-1} \times \underline{X^{i-1}}\,*}\left(w^{i-1}_{m-s-k} \cdot \sigma^k_{i-1} \cdot z^{i-1}_{n-i+1-j} \cdot \eta_{i-1}\right) \times h^s,$$

with $\sigma^k_{i-1} = \pi_{(\underline{i-1},i)\,*} \circ \pi^*_{(i-1,\underline{i})}(z^i_{n-2i+k}) \in Ch^j(G_{i-1\,K})$.

3. Equivalence

In this section, we continue to use notation and material introduced in the previous sections and we prove Theorem 1.2.

For $1 \leq i \leq d$ and $0 \leq j \leq i-1$, we set

$$\alpha_{i,j} := (\theta_i)_*(Z^i_{n-i-j}) + \rho_{i,j} \in \mathrm{CH}\left(X_K^{i+1}\right),$$

and we view the cycle $\alpha_{i,j}$ as a correspondence $X_K \rightsquigarrow X^i_K$.