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SOME REMARKS ON THE OPTIMALITY OF THE BRUNO-RÜSSMANN CONDITION

by Abed Bounemoura

Abstract. — We prove that the Bruno-Rüssmann condition is optimal for the analytic preservation of a quasi-periodic invariant curve for an analytic twist map. The proof is based on Yoccoz's corresponding result for analytic circle diffeomorphisms and the uniqueness of invariant curves with a given irrational rotation number. We also prove a similar result for analytic Tonelli Hamiltonian flow with $n = 2$ degrees of freedom; for $n \geq 3$ we only obtain a weaker result which recovers and slightly improves a theorem of Bessi.

Résumé (*Quelques remarques sur l'optimalité de la condition de Bruno-Rüssmann*). — Nous montrons que la condition de Bruno-Rüssmann est optimale pour la persistance de courbe invariante quasi-périodique analytique par une application twist analytique. La preuve repose sur le résultat analogue de Yoccoz pour un difféomorphisme analytique du cercle et sur l'unicité des courbes invariantes de nombre de rotation irrationnel. Nous montrons également un résultat similaire pour les Hamiltoniens Tonelli à $n = 2$ degrés de liberté; pour $n \geq 3$, nous obtenons un résultat plus faible qui généralise légèrement un théorème de Bessi.

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Abed Bounemoura, CNRS – PSL Research University, (Université Paris-Dauphine and Observatoire de Paris)

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1. Introduction

Given $n \geq 2$, a vector $\omega \in \mathbb{R}^n$ satisfies the Bruno-Rüssmann condition, and we will write $\omega \in \mathrm{BR}$, if

$$\int_1^{+\infty} \frac{\ln(\Psi_\omega(Q))}{Q^2} dQ < +\infty \tag{BR}$$

where

$$\Psi_\omega(Q) = \max\left\{|k \cdot \omega|^{-1} \mid k \in \mathbb{Z}^n,\ 0 < |k| \leq Q\right\}.$$

The expression in (BR) is just one of the many equivalent ways of defining this Bruno-Rüssmann condition. Bruno [2, 3, 4] and Rüssmann [20, 21, 22, 23] have proved that $\omega \in \mathrm{BR}$ is a sufficient condition for several analytic small divisors problems: among others, for the linearization of a holomorphic germ at a non-resonant fixed point, for the linearization of a torus diffeomorphism isotopic to the identity (respectively a torus vector field) close to a non-resonant translation (respectively close to a non-resonant constant vector field), and for the preservation of a non-resonant quasi-periodic invariant torus in a non-degenerate Hamiltonian system close to being integrable.

For $n = 2$, $\omega = (1, \alpha) \in \mathrm{BR}$ if and only if α satisfies the following Bruno condition, that we shall write $\alpha \in \mathrm{B}$:

$$\sum_{n \in \mathbb{N}} \frac{\log q_{n+1}}{q_n} < +\infty \tag{B}$$

where q_n is the denominator of the n^{th}-convergent of α. A major finding of Yoccoz ([27], [28]) is that if $\alpha \notin \mathrm{B}$, then the quadratic polynomial

$$P_\lambda(z) = \lambda z + z^2, \quad \lambda = e^{2\pi i \alpha}$$

is not analytically linearizable. Other examples of non-Bruno non-linearizable germs were later given by Geyer [10]. Using this, Yoccoz was able to prove that if $\alpha \notin B$, there exists, arbitrarily close to the rotation α, analytic circle diffeomorphisms which are topologically but not analytically conjugate to α and thus in the continuous case, if $\omega \notin \mathrm{BR}$, there exist, arbitrarily close to the constant vector field ω, analytic vector fields on $\mathbb{T}^2$ that are topologically but not analytically conjugate to ω (see Theorems 2.2 and 3.2 below for more precise statements). The condition $\alpha \in \mathrm{B}$ (or equivalently $\omega \in \mathrm{BR}$) is also known to be optimal in other problems in $\mathbb{C}^2$, for vector fields close to a non-resonant singular point [18] and for the complex area-preserving map known as the semi-standard map [14].

Unfortunately, to the best of our knowledge, the Bruno-Rüssmann condition is not known to be optimal for low-dimensional Hamiltonian problems such as the analytic preservation of invariant curves for twist maps. Here it is important to point out that unlike the other problems we mentioned which deal only with the existence of an analytic conjugacy to the linear model, in the Hamiltonian case the conclusions of KAM-like theorems are two-fold: it gives the existence of an analytic invariant curve together with the existence of an analytic conjugacy of the restricted dynamics on the curve to the linear model. The best known result for twist maps is due to Forni [8][1]. To describe his result, let us first remark that $\alpha \in \mathrm{B}$ obviously implies that $\alpha \in \mathrm{R}$ in the sense that

$$\text{(R)} \qquad \lim_{n\to+\infty} \frac{\log q_{n+1}}{q_n} = 0$$

but clearly the converse is not true. The condition that $\alpha \in \mathrm{R}$ is in fact the necessary and sufficient condition for the linearized problem (the so-called cohomological equation) to have a solution in the analytic topology [19]. Using results of Mather [15, 16] and Herman [12], Forni proved that if an integrable twist map has an invariant curve with rotation number $\alpha \notin \mathrm{R}$, then there exists an arbitrarily small analytic perturbation for which there are no (necessarily Lipschitz) invariant curves with rotation number α. In the case of a not necessarily integrable twist map, the conclusion remains true but under the stronger assumption that

$$\lim_{n\to+\infty} \frac{\log\log q_{n+1}}{q_n} = 0.$$

Observe that this strongly violates the conclusion of the KAM theorem, as the latter would give an analytic invariant curve on which the dynamic is analytically linearizable. For Tonelli Hamiltonian flows close to integrable with $n \geq 2$ degrees of freedom, a result analogous to Forni's has been obtained by Bessi [1]. To state it, observe that a generalization of the condition $\alpha \in \mathrm{R}$ is (keeping the same notation) $\omega \in \mathrm{R}$ where

$$\text{(R)} \qquad \lim_{Q\to+\infty} \frac{\ln(\Psi_\omega(Q))}{Q} = 0$$

and that again this is the necessary and sufficient condition to solve the cohomological equation in the analytic topology. Bessi proved that if $\omega \notin \mathrm{R}$, then there exists an arbitrarily small perturbation of the integrable Hamiltonian $H_0(I) = \frac{1}{2}(I_1^2 + \cdots + I_n^2)$ in the analytic topology for which there is no invariant C^1 Lagrangian graph on which the dynamic is C^1 conjugated to the linear flow of frequency ω.

The purpose of this note is to prove that the condition $\alpha \in \mathrm{B}$ is optimal for the analytic KAM theorem for twist maps, in the sense that if $\alpha \notin \mathrm{B}$, then there exists arbitrarily small perturbations of an arbitrary twist map for which there are no analytic invariant curves on which the dynamic is analytically conjugated

1. Unfortunately, at several places in the literature (for instance [9] and other references therein by the same author) it is stated that $\alpha \in \mathrm{B}$ is optimal for the existence of an analytic invariant circle for the standard map in the perturbative regime which depends analytically on the small parameter; we would like to point out that this statement is incorrectly deduced from results of Marmi [14] and Davie [5], and thus the optimality of $\alpha \in \mathrm{B}$ for the standard map is still an open question (see [17] where this observation is also made).

to α. We refer to Theorem A for a more precise statement. One has to observe that this result does not improve Forni's result, as in our example, the perturbed map will have an analytic invariant curve on which the dynamic is topologically conjugated to α, yet there will be no analytic conjugacy and this is sufficient to guarantee that the conclusions of the KAM theorem do not hold. One can considered Forni's result as a "destruction" of an invariant circle with rotation number $\alpha \notin \mathrm{R}$, while our result can be considered as a "destruction" of the dynamic on the invariant circle with rotation number $\alpha \notin \mathrm{B}$. For perturbations of Tonelli Hamiltonians, we will obtain in Theorem B a similar result showing the optimality of $\omega \in \mathrm{BR}$ for $n = 2$ while for $n \geq 3$, we will only obtain in Theorem C a result similar to Bessi showing that $\omega \in \mathrm{R}$ is necessary: for $n \geq 3$, it is unlikely that $\omega \in \mathrm{R}$ is sufficient and one should not expect $\omega \in \mathrm{BR}$ to be necessary either[2]. Even though we will use the action-minimizing properties of invariant quasi-periodic curves and tori in an indirect way, our method of proof is very different from those of Forni and Bessi. For Theorem A, we will use Yoccoz's result showing the necessity of $\alpha \in \mathrm{B}$ for the analytic linearization of circle diffeomorphims, and the well-known fact that an invariant curve for a twist map with a given irrational rotation number is unique. Under some more assumptions, this uniqueness property has been shown to be true for Tonelli Hamiltonians in any number of degrees of freedom by Fathi, Giualiani and Sorrentino [6]. Using this and a continuous version of Yoccoz's result, we will obtain Theorem B for the case $n = 2$ and for $n \geq 3$, we will make use of a result of Fayad [7] on reparametrized linear flows to obtain Theorem C.

2. The case of a twist map

It will be more convenient for us to represent an exact area-preserving map of the annulus $\mathbb{T} \times \mathbb{R}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, by a "Hamiltonian" generating function defined on the universal cover $\mathbb{R}^2$ of $\mathbb{T} \times \mathbb{R}$ (unlike [8] where a "Lagrangian" generating function is used). Given a smooth function $h : \mathbb{R}^2 \to \mathbb{R}$ such that $h(\theta + 1, \mathcal{I}) = h(\theta, \mathcal{I})$, the map

$$\bar{f} = \bar{f}_h : \mathbb{R}^2 \to \mathbb{R}^2$$

defined by

$$\bar{f}(\theta, I) = (\Theta, \mathcal{I}) \Longleftrightarrow \begin{cases} \mathcal{I} = I - \partial_\theta h(\theta, \mathcal{I}), \\ \Theta = \theta + \partial_{\mathcal{I}} h(\theta, \mathcal{I}) \end{cases}$$

projects to an exact area-preserving map

$$f : \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}.$$

Such a map is an exact area-preserving twist map, or for short a twist map in the sequel, if it satisfies the following two conditions:

2. Yoccoz, private communication.

(a1) for all $(\theta, I) \in \mathbb{R}^2$, $\partial_I \Theta(\theta, I) > 0$;
(a2) for all $\theta \in \mathbb{R}$, $|\Theta| \to +\infty$ as $|I| \to +\infty$ uniformly in θ.

Given such a twist map f, an invariant curve T for f will be an essential topological circle such that $f(T) = T$; necessarily, T is a Lipschitz Lagrangian graph. Let us denote by $\bar{T}$ the lift of such an invariant curve to the universal cover $\mathbb{R}^2$; the restriction $\bar{f}_{|\bar{T}}$ (which is the lift of the orientation-preserving circle diffeomorphism $f_{|T}$) has a well-defined rotation number in $\mathbb{R}$. The following uniqueness result is well-known (see [11] for instance).

Proposition 2.1. — *Let T_0 and T_1 be two invariant curves for a twist map f such that $\bar{f}_{|\bar{T}_0}$ and $\bar{f}_{|\bar{T}_1}$ have the same irrational rotation number. Then $T_0 = T_1$.*

Now let us explain the local setting in which the KAM theorem applies. Consider a smooth function $h_0 : \mathbb{R} \times (-1, 1) \to \mathbb{R}$ and its associated exact area-preserving map f_0 for which the following conditions are satisfied:

(b1) f_0 satisfies (a1) on $\mathbb{R} \times (-1, 1)$;
(b2) $\partial_\theta h_0(\theta, 0) = 0$ and $\partial_{\mathcal{I}} h_0(\theta, 0) = \alpha$.

If follows from (b2) that the map f_0 leaves the curve $T_0 = \mathbb{T} \times \{0\}$ invariant, and the restriction of $\bar{f}_0$ to $\bar{T}_0$ is the rotation by α. To state the KAM theorem of Bruno and Rüssmann, we need to define norms for real-analytic functions. Let $h : \mathbb{R} \times (-1, 1) \to \mathbb{R}$ be a real-analytic function and suppose it admits a holomorphic and bounded extension (still denoted by h) to the domain

$$\mathbb{T}_s \times D = \left\{ z = (z_1, z_2) \in \mathbb{C}^2 \mid |\mathrm{Im}\, z_1| < s,\ |z_2| < 1 \right\}$$

for some $s > 0$. In such a case, we simply define

$$|h|_s = \sup_{z \in \mathbb{T}_s \times D} |h(z)|.$$

Assume that h_0 satisfies condition (b1) and (b2) with $\alpha \in \mathrm{B}$, then the KAM theorem states that for any $s > 0$, there exists $\varepsilon > 0$ such that for any h_1 satisfying $|h_1 - h_0|_s \leq \varepsilon$, the exact area-preserving map f_1 generated by h_1 has an analytic invariant curve T_1 such that $\bar{f}_{1|\bar{T}_1}$ is analytically conjugated (by the lift of an orientation preserving circle diffeomorphism) to the rotation α (and moreover, T_1 analytically converges to T_0 as ε goes to zero).

The following result shows that the condition that $\alpha \in \mathrm{B}$ cannot be weakened.

Theorem A. — *Assume that h_0 satisfies conditions (b1) and (b2) with $\alpha \notin \mathrm{B}$. Then for all $\varepsilon > 0$ sufficiently small and all $s > 0$, there exists h_1 such that $|h_1 - h_0|_s \leq \varepsilon$ and the exact area-preserving map f_1 generated by h_1 has no analytic invariant curve T_1 such that $\bar{f}_{1|\bar{T}_1}$ is analytically conjugated (by the lift of an orientation preserving circle diffeomorphism) to the rotation α.*