ISSN 0012-9593

quatrième série - tome 52 fascicule 1 janvier-février 2019

ANNALES SCIENTIFIQUES de L'ÉCOLE NORMALE SUPÉRIEURE

Davoud CHERAGHI

Typical orbits of quadratic polynomials with a neutral fixed point: Non-Brjuno type

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Annales Scientifiques de l'École Normale Supérieure

Publiées avec le concours du Centre National de la Recherche Scientifique

Rédaction / *Editor*

Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

Édition et abonnements / *Publication and subscriptions*

Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

Tarifs

Abonnement électronique : 420 euros.
Abonnement avec supplément papier :
Europe : 551 €. Hors Europe : 620 € ($ 930). Vente au numéro : 77 €.

ISSN 0012-9593 (print) 1873-2151 (electronic)

Directeur de la publication : Stéphane Seuret
Périodicité : 6 nos / an

Ann. Scient. Éc. Norm. Sup.
4[e] série, t. 52, 2019, p. 59 à 138

TYPICAL ORBITS OF QUADRATIC POLYNOMIALS WITH A NEUTRAL FIXED POINT: NON-BRJUNO TYPE

BY DAVOUD CHERAGHI

ABSTRACT. – We investigate the quantitative and analytic aspects of the near-parabolic renormalization scheme introduced by Inou and Shishikura in 2006. These provide techniques to study the dynamics of some holomorphic maps of the form $f(z) = e^{2\pi i\alpha} z + \mathcal{O}(z^2)$, including the quadratic polynomials $e^{2\pi i\alpha} z + z^2$, for some irrational values of α. The main results of the paper concern fine-scale features of the measure-theoretic attractors of these maps, and their dependence on the data. As a bi-product, we establish an optimal upper bound on the size of the maximal linearization domain in terms of the Siegel-Brjuno-Yoccoz series of α.

RÉSUMÉ. – On étudie les aspects quantitatifs et analytiques du procédé de renormalisation presque parabolique introduit par Inou et Shishikura en 2006. Ceci fournit des techniques pour étudier la dynamique de certaines applications holomorphes de la forme $f(z) = e^{2\pi i\alpha} z + \mathcal{O}(z^2)$, dont les polynômes quadratiques $e^{2\pi i\alpha} z + z^2$, pour certaines valeurs irrationnelles de α. Les principaux résultats de cet article concernent les propriétés à petite échelle des attracteurs au sens de la théorie de la mesure pour ces applications ainsi que de leur dépendance en fonction des données du problème. On obtient également une borne supérieure optimale sur la taille du domaine maximal de linéarisation en termes de la série de Brjuno-Siegel-Yoccoz de α.

1. Introduction

1.1. Neutral fixed points

Let f be a holomorphic map of the form

$$f(z) = e^{2\pi i\alpha} z + a_2 z^2 + a_3 z^3 + \cdots ,$$

defined on a neighborhood of $0 \in \mathbb{C}$, and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Asymptotically near 0, the orbits are governed by the rotation of angle α and are highly recurrent. Away from zero, the influence of non-linearity increases, eventually reaching the scale where the behavior is governed by the global topological structure of the map. For systems with unstable behavior near zero, the transition from local to global and back may occur infinitely often. This creates a delicate

doi:10.24033/asens.2384

interplay among the arithmetic nature of α, the non-linearities of the large iterates of f, and the global covering structure of the iterates of f. In this paper we study this problem.

The ideal scenario is when the map f is conformally conjugate to the linear map $w \mapsto e^{2\pi \mathbf{i}\alpha} \cdot w$ on a neighborhood of 0. To discuss this further, let us denote the best rational approximants of α with p_n/q_n, $n \geq 1$. By a landmark result of Siegel and Brjuno [38, 5], if the series

$$\mathcal{B}(\alpha) = \sum_{n=1}^{\infty} q_n^{-1} \log q_{n+1}$$

is finite, then near 0 the map f is conformally conjugate to a linear map. When f is linearizable near 0, the maximal domain on which the conjugacy exists is called the *Siegel disk* of f. The geometry of the Siegel disks as well as the dynamics of f near their boundaries have been the subject of extensive studies over the last few decades. These involve a wide range of methods, with consequences often depending on the arithmetic nature of α. See for instance, [22, 28, 20, 21, 33, 1, 7, 40, 42, 43, 12]. We note that for almost every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{B}(\alpha) < \infty$, while for generic choice of $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\mathcal{B}(\alpha) = \infty$.

On the other hand, by a celebrated result of Yoccoz [41], if $\mathcal{B}(\alpha) = \infty$, the polynomial

$$P_\alpha(z) := e^{2\pi \mathbf{i}\alpha} z + z^2$$

is not linearizable at zero. Although this optimality result has been further extended by similar ideas to special families of maps [31, 19, 29], it remains widely open in families of polynomials and rational maps. Also, due to a non dynamical step in those arguments, very little has been understood about the local dynamics of non-linearizable maps. In [32], Perez-Marco constructs non-trivial local invariant compact sets containing 0 for non-linearizable maps. But the necessary control on the geometry of these objects and the dynamics of the map on them has remained out of reach.

In 2006, Inou and Shishikura introduced a renormalization scheme that provides a powerful tool to study the dynamics of near parabolic maps, [24]. This involves an infinite-dimensional class of maps $\mathcal{F}$, and a nonlinear operator $\mathcal{R} : \mathcal{F} \to \mathcal{F}$, called near-parabolic renormalization. Every map in $\mathcal{F}$ is defined on a Jordan neighborhood of 0, has a neutral fixed point at 0, and a unique critical point of local degree two in its domain of definition. Given $f \in \mathcal{F}$, $\mathcal{R}(f)$ is defined as a sophisticated notion of the return map of f about 0 to a region in the domain of f, viewed in a certain canonically defined coordinate on that region. Precise definitions appear in Section 2.

In this paper we carry out a quantitative analysis of the near-parabolic renormalization scheme. This involves proving a number of foundational results on the combinatorial and analytic aspects of the scheme. In particular, we have slightly modified the definition of renormalization to make it suitable for applications.

Successive iterates of $\mathcal{R}$ at some $f \in \mathcal{F}$ produce a renormalization tower; a sequence of maps $\mathcal{R}^{\circ j}(f)$ which are related by the changes of coordinates. The general theme in theories of renormalization is that large iterates of f often break down into compositions of a small number of the changes of coordinates and the maps $\mathcal{R}^{\circ j}(f)$. However, due to the "semi-local" nature of near-parabolic renormalization, there are a number of issues which require careful consideration.

The maps in $\mathcal{F}$ have a partial covering structure, involving a branched covering of local degree two. The change of coordinate in the definition of renormalization also has a partial covering structure with a branch point. We prove some detailed orbit relations on the renormalization tower, relating the combinatorial aspects of the orbits of f to the ones of $\mathcal{R}^{\circ j}(f)$, for $j \geq 1$.

The change of coordinates in the definition of renormalization involves transcendental mappings with highly distorting nature. Substantial part of the paper (Section 6) is devoted to proving uniform (distortion) estimates on these maps and their dependence on α. To this end, we have introduced a new approach to compare the changes of coordinates to some model maps using quasi-conformal mappings.

The interplay between the arithmetic of α and the non-linearities of the iterates of f is manifested in geometric aspects of the renormalization tower. We present a systematic approach to employing the above combinatorial and analytic tools to study the dynamics of the maps in $\mathcal{F}$.

1.2. Statements of the results

Define the set of irrational numbers

$$\mathrm{HT}_N := \{[0; a_1, a_2, \dots] \in \mathbb{R} \mid \forall i \geq 1, a_i \geq N\},$$

where $N \in \mathbb{N}$ and $[0; a_1, a_2, \dots] = 1/(a_1 + 1/(a_2 + 1/(\cdots)))$ denotes the continued fraction expansion. For technical reasons, in this paper we require α to be in HT_N, for some fixed constant $N \in \mathbb{N}$. (1)

The class of maps $\mathcal{F}$ fibers over HT_N as

$$\mathcal{F} = \bigcup_{\alpha \in \mathrm{HT}_N} \mathcal{F}_\alpha, \quad \mathcal{F}_\alpha = \{P_\alpha\} \cup \mathcal{IS}_\alpha,$$

where $\mathcal{IS}_\alpha$ is the Inou-Shishikura class of maps defined precisely in Section 2.2. For $f \in \mathcal{IS}_\alpha$, $f'(0) = e^{2\pi \mathbf{i} \alpha}$. We note that for each $\alpha \in \mathrm{HT}_N$ there are polynomials and rational maps of arbitrarily large degree whose restriction to some neighborhood of 0 belongs to $\mathcal{IS}_\alpha$. Also, $P_\alpha \notin \mathcal{IS}_\alpha$, but $\mathcal{R}(P_\alpha)$ is defined and belongs to $\mathcal{IS}_{1/\alpha} \subset \mathcal{F}$.

By classical results, the post-critical set of a holomorphic map provides key information about the dynamics of that map, in particular, its measurable dynamics. A map $f \in \mathcal{F}$ has a unique critical point in its (restricted) domain of definition, say cp_f. The *post-critical* set of f associated to cp_f is defined as

$$\mathcal{PC}(f) = \overline{\bigcup_{i=1}^{\infty} f^{\circ i}(\mathrm{cp}_f)}.$$

The main aim of this paper is to describe the geometry of the post-critical set and the iterates of the map near it. To this end, we build a decreasing nest of simply connected sets containing $\mathcal{PC}(f)$, denoted by Ω_0^n, $n \geq 0$. Each Ω_0^n is formed of about $q_{n+1} + q_n$ (topological) sectors landing at 0, which are ordered by the arithmetic of α, and are mapped to one another by the map. Roughly speaking, the rotation element leads to a tangential action on each Ω_0^n, while the nonlinearity of the map results in a radial action on each Ω_0^n.

(1) This is also required in the near-parabolic renormalization scheme. However, it is conjectured that there exists a scheme with similar qualitative features for which $N = 1$. So, we hope that the arguments presented here will be eventually applied to all irrational rotation numbers.