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LONG-TIME HOMOGENIZATION AND ASYMPTOTIC BALLISTIC TRANSPORT OF CLASSICAL WAVES

BY Antoine BENOIT AND Antoine GLORIA

ABSTRACT. – Consider an elliptic operator in divergence form with symmetric coefficients. If the diffusion coefficients are periodic, the Bloch theorem allows one to diagonalize the elliptic operator, which is key to the spectral properties of the elliptic operator and the usual starting point for the study of its long-time homogenization. When the coefficients are not periodic (say, quasi-periodic, almost periodic, or random with decaying correlations at infinity), the Bloch theorem does not hold and both the spectral properties and the long-time behavior of the associated operator are unclear. At low frequencies, we may however consider a formal Taylor expansion of Bloch waves (whether they exist or not) based on correctors in elliptic homogenization. The associated Taylor-Bloch waves diagonalize the elliptic operator up to an error term (an "eigendefect"), which we express with the help of a new family of extended correctors. We use the Taylor-Bloch waves with eigendefects to quantify the transport properties and homogenization error over large times for the wave equation in terms of the spatial growth of these extended correctors. On the one hand, this quantifies the validity of homogenization over large times (both for the standard homogenized equation and higher-order versions). On the other hand, this allows us to prove asymptotic ballistic transport of classical waves at low energies for almost periodic and random operators.

RÉSUMÉ. – Considérons un opérateur elliptique sous forme divergence à coefficients symétriques non constants. Si ces coefficients sont périodiques, la théorie de Floquet-Bloch permet de diagonaliser l'opérateur elliptique, ce qui est crucial pour l'étude des propriétés spectrales de l'opérateur et le point de départ usuel pour l'étude des propriétés d'homogénéisation en temps long de l'opérateur des ondes associé. Quand les coefficients ne sont pas périodiques (disons quasi-périodiques, presque périodiques, ou aléatoires stationnaires ergodiques), la théorie de Floquet-Bloch ne s'applique plus et les propriétés spectrales ainsi que le comportement en temps long de l'opérateur des ondes associé ne sont pas claires *a priori*. Aux basses fréquences, nous pouvons cependant considérer un développement de Taylor formel des ondes de Bloch (que celles-ci existent ou non) en se basant sur des correcteurs introduits en homogénéisation elliptique. Ces ondes de Taylor-Bloch diagonalisent l'opérateur elliptique à un terme d'erreur près (un "défaut propre"), que nous exprimons à l'aide d'une nouvelle famille de correcteurs étendus. Nous utilisons cette formulation des défauts propres pour quantifier les propriétés de transport et d'homogénéisation en temps long pour l'équation des ondes associée en termes de croissance spatiale des correcteurs étendus. D'une part, cela quantifie la validité de l'homogénéisation en temps long (à la fois pour l'opérateur homogénéisé standard et pour des versions d'ordre supérieur).

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D'autre part, cela nous permet d'établir le transport balistique asymptotique des ondes classiques aux basses énergies pour des opérateurs presque périodiques et aléatoires.

1. Introduction

Let $\boldsymbol{a}$ be a periodic symmetric coefficient field, and consider the rescaled wave operator $\Box_\varepsilon := \partial_{tt}^2 - \nabla \cdot \boldsymbol{a}(\frac{\cdot}{\varepsilon})\nabla$. It is known since the pioneering works in homogenization that for fixed final time $T < \infty$, the operator $\Box_\varepsilon$ can be replaced by the homogenized wave operator $\Box_{\mathrm{hom}} := \partial_{tt}^2 - \nabla \cdot \boldsymbol{a}_{\mathrm{hom}}\nabla$, where $\boldsymbol{a}_{\mathrm{hom}}$ are the homogenized (and constant) coefficients associated with $\boldsymbol{a}$ through elliptic homogenization (see [13, 23] where the question of the corrector and convergence of the energy is also addressed, and Section 2 for precise definitions). Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz class (most of the results of this article hold for initial conditions in some Hilbert space $H^m(\mathbb{R}^d)$ for m large enough), let $u_\varepsilon \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ be the unique weak solution of

$$\begin{cases} \Box_\varepsilon u_\varepsilon(t,x) = 0, \\ u_\varepsilon(0,\cdot) = u_0, \\ \partial_t u_\varepsilon(0,\cdot) = 0. \end{cases} \tag{1.1}$$

Then for all $T > 0$, $\lim_{\varepsilon\downarrow 0} \sup_{0\le t\le T} \|u_\varepsilon(t,\cdot) - u_{\mathrm{hom}}(t,\cdot)\|_{L^2(\mathbb{R}^d)} = 0$, where u_{hom} solves the homogenized equation

$$\begin{cases} \Box_{\mathrm{hom}} u_{\mathrm{hom}}(t,x) = 0, \\ u_{\mathrm{hom}}(0,\cdot) = u_0, \\ \partial_t u_{\mathrm{hom}}(0,\cdot) = 0. \end{cases} \tag{1.2}$$

Refining this result received much attention in the recent years—and in particular the large-time behavior of u_ε with fixed or oscillating initial conditions. For fixed initial conditions u_0 (independent of ε), one expects dispersive effects—which are not accounted for by (1.2)—to appear at times of order $\varepsilon^{-2}T$ (see [38] for pioneering works in this direction, [17, 18, 34] for the first rigorous results, and [1, 2] for numerical methods). For oscillating initial conditions, the medium interacts with the initial conditions much more, which yields even finer dispersive effects (see [7, 8]). Both refinements crucially rely on spectral properties of the operator $-\nabla \cdot \boldsymbol{a}\nabla$, namely that it is diagonalized by Floquet-Bloch waves (Bloch in short, see [6, 5, 7, 8, 17, 18]): the spectrum of $-\nabla \cdot \boldsymbol{a}\nabla$ is purely absolutely continuous, and extended states are semi-explicit (see below). Hence, there is a clear starting point to study the above questions: project the initial condition on the Bloch wave basis, and treat the wave Equation (1.1) as an ODE. From a spectral point of view, the Bloch theory implies that $-\nabla \cdot \boldsymbol{a}\nabla$ has purely absolutely continuous spectrum in form of possibly overlapping bands (the first one including 0).

The Bloch theory crucially relies on the periodicity of $\boldsymbol{a}$, and can be seen as a variant of the Fourier transform (with which it coincides when $\boldsymbol{a}$ is a constant matrix). The main idea is to look for extended states of the operator $-\nabla \cdot \boldsymbol{a}\nabla$ in the form of modulated plane waves

$x \mapsto e^{ik\cdot x}\psi_k(x)$, where ψ_k is a periodic function. Such a function ψ_k is then solution of the magnetic eigenvalue problem on the torus

$$-(\nabla + ik)\cdot \boldsymbol{a}(\nabla + ik)\psi_k = \lambda_k \psi_k,$$

for some λ_k. By the Rellich theorem, $-(\nabla + ik) \cdot \boldsymbol{a}(\nabla + ik)$ has compact resolvent, which allows one to define a family of eigenvectors and eigenvalues, on which the Bloch decomposition relies. Replace $\boldsymbol{a}$ by the sum of two periodic functions with incommensurable periods, and the whole picture breaks down: the magnetic operator is now lifted to a higher-dimensional torus, it is hypo-elliptic, and does not have compact resolvent any longer. In particular, we do not know whether the ψ_k exist. For more general coefficients (say almost-periodic, or random), the Bloch theory simply does not hold. Indeed, this theory implies that the elliptic operator $-\nabla\cdot\boldsymbol{a}\nabla$ has purely absolutely continuous spectrum, whereas it is known that this operator has some discrete spectrum in any dimension for some representative examples of $\boldsymbol{a}$, cf. [39, Theorem 3.3.6].

Questions regarding oscillating initial data explore the entire spectrum of the operator $-\nabla \cdot \boldsymbol{a}\nabla$, and we expect a completely different behavior for periodic and non-periodic coefficients, since their spectrum is of different type. This is the realm of challenging questions of spectral analysis [39, 3] and radiative transport [36, 37]. For non-oscillating initial data however, only the bottom of the spectrum of $-\nabla \cdot \boldsymbol{a}\nabla$ is relevant, and we are in the realm of homogenization. For final times $T < \infty$ independent of ε, qualitative theory for the elliptic operator is enough to prove the convergence of (1.1) to (1.2), and we just need to know that the solution operator $(-\nabla \cdot \boldsymbol{a}(\frac{\cdot}{\varepsilon})\nabla)^{-1}$ converges to the homogenized solution operator $(-\nabla \cdot \boldsymbol{a}_{\mathrm{hom}}\nabla)^{-1}$ as $\varepsilon \downarrow 0$. If we manage to have quantitative information on this convergence in terms of ε (in a broad sense), we might be able to consider larger time frames $[0, \varepsilon^{-\alpha}T]$ (with $\alpha > 0$) and gain information on the large-time behavior of u_ε. The aim of this contribution is to develop such an approach for operators that are beyond the reach of the classical Bloch theory.

As a first and crucial step, we introduce a proxy for the Bloch waves decomposition. Since we are only interested in low frequencies, we only need a proxy for Bloch waves at low frequencies. In the case of periodic coefficients, it is well-known that Bloch waves ψ_k are essentially analytic functions of k, and that their derivatives are related to cell-problems in elliptic homogenization (e.g., [16, 4]). Whereas eigenvectors ψ_k might not exist (even at low frequencies), one may still consider their formal Taylor expansion $\psi_{k,j}$ of order j for all $0 \le |k| \ll 1$ based on correctors (up to order j) provided the latter exist, which gives rise to what we call Taylor-Bloch waves $x \mapsto e^{ik\cdot x}\psi_{k,j}(x)$. These waves are only "approximate" extended states of the operator $-\nabla \cdot \boldsymbol{a}\nabla$, so that the study of the defect in the eigenvector/eigenvalue relation (which we call the "eigendefect") is equally important as the formula for the Taylor-Bloch waves itself. The study of the Taylor-Bloch expansion is the aim of Section 2, where we introduce a new family of higher-order correctors, that are used to put the eigendefect in a suitable form for the rest of our analysis.

The second step consists in constructing an approximate solution to Equation (1.1) using the Taylor-Bloch waves, cf. Section 3. We first replace the initial condition by a well-prepared initial condition in the form of a Taylor-Bloch expansion—which simply amounts