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RENORMALIZATION IN QUANTUM FIELD THEORY (AFTER R. BORCHERDS)

par Estanislao HERSCOVICH

Abstract.——The aim of this manuscript is to provide a complete and precise formulation of the renormalization picture for perturbative Quantum Field Theory (pQFT) on general curved spacetimes introduced by R. Borcherds in [**10**]. More precisely, we give a full proof of the free and transitive action of the group of renormalizations on the set of Feynman measures associated with a local precut propagator, and that such a set is nonempty if the propagator is further assumed to be manageable and of cut type. Even though we follow the general principles laid by Borcherds in [**10**], we have in many cases proceeded differently to prove his claims, and we have also needed to add some hypotheses to be able to prove the corresponding statements.

Résumé. **(Rénormalisation en théorie quantique de champs (d'après R. Borcherds))**——Le but de ce manuscrit est de présenter une formulation complète et précise de la procédure de renormalisation dans la théorie quantique des champs perturbative (pQFT) sur des espaces-temps courbes, introduite par R. Borcherds dans [**10**]. Plus précisément, nous donnons une preuve exhaustive de l'action simplement transitive du groupe de renormalisation sur l'ensemble des mesures de Feynman associées à un propagateur de type coupe, et sur l'existence des mesures de Feynman associées à tels propagateurs. Même si nous avons suivi les principes généraux posés par Borcherds dans [**10**], souvent nous avons procédé différemment pour prouver ses affirmations, et nous avons dû aussi ajouter quelques hypothèses pour pouvoir démontrer les assertions respectives.

CONTENTS

INTRODUCTION

General description

The aim of this work is twofold:

(i) To give a complete and precise formulation of the renormalization picture for perturbative Quantum Field Theory (pQFT), following the point of view introduced by R. Borcherds in [**10**]. In particular, we explain in full detail the different objects he introduces, together with their algebraic and topological structures.

(ii) To give a complete proof of Theorems 15, 18, 20 and 21 in [**10**] about the free and transitive action of the group of renormalizations on the set of Feynman measures associated with a local propagator of cut type, as well as the existence of a Feynman measure associated with any such propagator. This is done in Theorems 6.0.8, 7.3.10, 7.4.2 and 7.5.2. For the existence we were obliged to impose a further condition on the propagator that does not appear in [**10**] and that we called *manageability* (see Definition 7.3.4). Without it we were unable to construct the needed Feynman measure, and we remark that this extra assumption is verified in the basic examples of scalar field theory or Dirac field theory on Minkowski, de Sitter and anti-de Sitter spacetime (see Remark 7.3.5).

Let us explain our motivations. The article [**10**] is really beautiful and full of ideas. However, it is very hard to read, because in many parts it seems that the author has chosen to simplify or to avoid the corresponding explanations, and in others there are several inaccuracies or potentially misleading statements. For example, the author never gives a precise definition of the *support* of an element in $S\Gamma_c\omega SJ\Phi$ (where we follow the notation in [**10**]), and even if one may assume that the support of a homogeneous element $A \in S^m\Gamma_c\omega SJ\Phi$ should be a subset of M^m, he talks about the intersection of supports of homogeneous elements of different degrees (e.g., see the proof of Thm. 15) or the causal comparison between both (e.g., see Def. 9). Another example of such imprecision is when he states that $S\Gamma_c\omega SJ\Phi$ is a comodule over $\Gamma SJ\Phi$ in Lemma 14, whereas the precise structure is never explained, or when he introduces the notion of Feynman measure in his Def. 9, in which he claims that there is a map $S^m\Gamma_c\omega SJ\Phi \to S^m\Gamma_c\omega SJ\Phi \otimes S^m\Gamma_c SJ\Phi$ induced by the coaction $\omega SJ\Phi \to \omega SJ\Phi \otimes SJ\Phi$. We claim that such structure does not exist in general, and no induced map canonically exists, respectively (see Section 3.5). In order to circumvent this problem, we had considered instead another comodule over the same coalgebra, which retains enough information about the space $S\Gamma_c\omega SJ\Phi$ (cf. Sections 3.7-3.12 and Lemma 5.8.7). We may also add that we have specified some (minor) missing hypotheses in **[10]**, such as the antisymmetric property for the causal relation on the definition of spacetime in his Def. 1, and we have made more precise the definition of cut propagator given in his Def. 7. See also Section 5.2 for some other technical differences with respect to the exposition of Borcherds. Despite of all of these facts, it is clear to us that all the statements in **[10]** are essentially correct. However, the technicality and the subtlety of the structures required to understand the objects presented in **[10]** is so involved, that we believe that the gaps left there are far from being automatically filled. Our humble intention is thus to clarify the technical details as well as the algebraic structures lurking behind to provide the corresponding precise statements with their proofs.

Let us also mention that even though our proofs are greatly inspired by those of **[10]**, we have in many cases proceeded in a different way. Compare for example the proofs of Lemma 14, and Thm. 15 in **[10]**, and those of Lemma 5.8.7 and Theorem 6.0.8. Concerning Thm. 18, 20 and 21 in **[10]**, we follow the general philosophy established there, but the actual proofs of those results, given in Theorems 7.3.10, 7.4.2 and 7.5.2 in this manuscript, respectively, are more involved.

Concerning the novelty of the results presented here, we believe that the contents of Chapter 1, 2, and 4 are mostly well-known to the specialists, but we provide them for the convenience of the reader, since they come from rather scattered fields. We may only except Sections 2.2 and 2.3 from the previous list, because they contain material that is new as far as we know. Corollaries 1.2.9 and 1.3.5, as well as Theorems 4.3.10 and 4.5.6, also seem to be new, even though the latter is an adaptation of a result of N. V. Dang in **[27]**. Chapter 3 is a generalization of some well-known algebraic structures for vector spaces to a more involved situation dealing with locally convex spaces (and in particular with sections of vector bundles), so it is in our opinion somehow new. Chapter 5 is essentially based on the article **[10]** of Borcherds, but we provide many explanations that are absent in the mentioned paper. In particular, Sections 5.3 and 5.9 are somehow implicit in **[10]**, as well as many results concerning propagators in Sections 5.4, 5.5 and 5.6. Moreover, since some of the results stated by Borcherds in **[10]** seem to be not completely correct or at least unclear (e.g., the existence of a coaction of $\Gamma SJ\Phi$ on the space of nonlocal Lagrangians $S\Gamma_c\omega SJ\Phi$ stated in his Lemma 14, or of the corresponding map $S^m\Gamma_c\omega SJ\Phi \to S^m\Gamma_c\omega SJ\Phi \otimes S^m\Gamma_c SJ\Phi$ induced by the coaction $\omega SJ\Phi \to \omega SJ\Phi \otimes SJ\Phi$ in his Def. 9 of Feynman measure), we deal in many situations with more involved algebraic structures than those in his exposition. For instance, we are forced to work with the tensor algebra instead of the symmetric algebra, and to show that (only) the final constructions of the theory depend on the class in the symmetric algebra of the considered elements. As stated before, this is however different from the considerations in **[10]**, but also from previous expositions, e.g., **[14]**, where the symmetric algebra was considered from the