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François LABOURIE & Richard WENTWORTH

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

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Société Mathématique de France
Institut Henri Poincaré
11, rue Pierre et Marie Curie
75231 Paris Cedex 05
Tél. : (33) 01 44 27 67 99
Fax : (33) 01 40 46 90 96

Abonnements / *Subscriptions*

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 09
Fax : (33) 04 91 41 17 51
email : `smf@smf.univ-mrs.fr`

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VARIATIONS ALONG THE FUCHSIAN LOCUS

BY François LABOURIE AND Richard WENTWORTH

ABSTRACT. – The main result is an explicit expression for the Pressure Metric on the Hitchin component of surface group representations into $\mathsf{PSL}(n,\mathbb{R})$ along the Fuchsian locus. The expression is in terms of a parametrization of the tangent space by holomorphic differentials, and it gives a precise relationship with the Petersson pairing. Along the way, variational formulas are established that generalize results from classical Teichmüller theory, such as Gardiner's formula, the relationship between length functions and Fenchel-Nielsen deformations, and variations of cross ratios.

RÉSUMÉ. – Notre résultat principal est une expression explicite de la métrique de pression sur la composante de Hitchin de l'espace des représentations du groupe fondamental d'une surface dans $\mathsf{PSL}(n,\mathbb{R})$ le long du lieu fuchsien. Cette formule utilise une paramétrisation de l'espace tangent à la composante de Hitchin en terme de différentielles holomorphes, et elle s'exprime explicitement en fonction du produit de Petersson. Au passage, nous établissons des relations qui généralisent les résultats classiques de la théorie de Teichmüller, tels que la formule de Gardiner, le rapport entre fonctions de longueur et déformations de Fenchel-Nielsen et les variations des birapports.

1. Introduction

Classical Teichmüller theory provides links between complex analytic and dynamical quantities defined on Riemann surfaces with conformal hyperbolic metrics. More precisely, properties of the geodesic flow of a hyperbolic structure are related to holomorphic objects on the underlying Riemann surface. The Selberg trace formula is an instance of this correspondence. The goal of this paper is to extend this relationship in the context of higher rank Teichmüller theory. Specifically, in the case of Hitchin representations we find analogs to the fundamental results of Wolpert—as well as those of Hejhal and Gardiner—that compute

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variations of dynamical quantities for deformations of the complex structure parametrized by holomorphic differentials. In particular, we refer here to Gardiner's formula [15] which computes the variation of the length of a geodesic in terms of Hejhal's periods of quadratic differentials; the relation between the Thurston and Weil-Petersson metrics [43]; the computation of the variation of the cross ratio on the boundary at infinity of surface groups and the study of Fenchel-Nielsen twists [42].

Let us be more concrete. Let X be a closed Riemann surface of genus at least two, and Σ the underlying oriented differentiable manifold. Let δ_X be the monodromy of the unique conformal hyperbolic metric on X. Let ι_n be the irreducible representation of $\mathsf{PSL}(2,\mathbb{R})$ in $\mathsf{PSL}(n,\mathbb{R})$. The *Fuchsian point* is the representation

$$\delta_{X,n} = \iota_n \circ \delta_X : \pi_1(\Sigma) \to \mathsf{PSL}(n,\mathbb{R}).$$

A *Hitchin representation* is a homomorphism $\delta : \pi_1(\Sigma) \to \mathsf{PSL}(n,\mathbb{R})$ that can be continuously deformed to the Fuchsian point. We call the set $\mathcal{H}(\Sigma,n)$ of conjugacy classes of Hitchin representations the *Hitchin component*. The *Fuchsian locus* is the subset of $\mathcal{H}(\Sigma,n)$ consisting of Fuchsian points obtained by varying the complex structure on X. By an abuse of terminology, we shall refer to these Fuchsian points as *Fuchsian representations* $\pi_1(\Sigma) \to \mathsf{PSL}(n,\mathbb{R})$. Furthermore, throughout this paper we can and will assume a lift of Hitchin representations from $\mathsf{PSL}(n,\mathbb{R})$ to $\mathsf{SL}(n,\mathbb{R})$.

Hitchin [22] proves that $\mathcal{H}(\Sigma,n)$ can be globally parametrized by the *Hitchin base*: $Q(X,n) = \bigoplus_{k=2}^{n} H^0(X,K^k)$, where K is the canonical bundle of X. Thus, the tangent space of the Fuchsian point of the Hitchin component can also be described as $Q(X,n)$. This infinitesimal parametrization, which will be crucial for our calculations, depends on some choices, and it is natural to normalize so that the restriction to the Fuchsian locus corresponds to classical deformations in Teichmüller space. In fact, in this paper we shall use two natural families of deformations (that is vectors in $\mathsf{T}_{\delta_{X,n}}\mathcal{H}(\Sigma,n)$ associated to a point $q \in Q(X,n)$) which are related by a constant depending only on n and k:

1. The *standard deformation* $\psi^0(q)$, for which the result of the computations is easier to state.
2. The *normalized deformation* $\psi(q)$, for which the Atiyah-Bott-Goldman symplectic structure of the Hitchin components coincides along the Fuchsian locus with the symplectic structure inherited from the L^2-metric on the Hitchin base (see Corollary 5.1.2).

Recall that the moduli space of representations $\pi_1(\Sigma) \to \mathsf{SL}(n,\mathbb{C})$ is a hyperkähler variety [21]. This structure is reflected in three algebraically distinct descriptions: the Dolbeault (Higgs bundle) moduli space, the de Rham moduli space of flat connections, and the Betti moduli space of representations. We exhibit isomorphisms of the tangent space to the Fuchsian point in each of these manifestations as $Q(X,n) \oplus \overline{Q(X,n)}$. We furthermore show that the different points of view actually give rise to the same parametrization of the tangent space at the Fuchsian point. A key point is that the first variation of the *harmonic* metric for certain variations of Higgs bundles vanishes (see Theorem 3.5.1). This result may be viewed as a generalization of Ahlfors' lemma on variations of the hyperbolic metric under quasiconformal deformations by harmonic Beltrami differentials [1]. All this occupies Section 3.

The discussion above is the complex analytic side of the Hitchin component, and we now wish to relate it to the dynamical side. In [26], the first author shows that if δ is a Hitchin representation and γ a nontrivial element in $\pi_1(\Sigma)$, then $\delta(\gamma)$ has n-distinct positive eigenvalues. The underlying idea is to associate to a Hitchin representation a *geodesic flow* (see also [19] and [8]), thus giving a dynamical characterization of the Hitchin component.

This leads to the main motivation for this paper. In [8], Bridgeman, Canary, Sambarino and the first author constructed a *pressure metric* on the Hitchin component whose restriction to the Fuchsian locus is the Weil-Petersson metric. In Section 6, we shall prove the following

THEOREM 1.0.1. – *Let δ be a Fuchsian representation into $\mathsf{SL}(n,\mathbb{R})$ associated to a Riemann surface X with a conformal hyperbolic metric. Let q be a holomorphic k-differential on X, $2 \leqslant k \leqslant n$, and let $\psi^0(q)$ be the associated standard deformation. Then the pressure metric is proportional to the L^2-metric:*

$$\mathbf{P}_\delta\left(\psi^0(q), \psi^0(q)\right) = \frac{1}{2^{k-1}\pi\,|\chi(X)|}\left[\frac{(k-1)!(n-1)!}{(n-k)!}\right]^2 \int_X \|q\|^2 d\sigma.$$

Moreover, two deformations associated to holomorphic differentials of different degrees are orthogonal with respect to the pressure metric.

The first ingredient in the proof of this theorem is an extension, Theorem 4.0.1, of Gardiner's formula to Hitchin representations. This computes the first variation of the eigenvalues of $\delta(\gamma)$ as a function of δ under a standard deformation. The result, proven in Section 4, is a generalization of the classical formula for holomorphic quadratic differentials [15]. We reproduce the statement here for the highest eigenvalue.

THEOREM 1.0.2 (Gardiner formula). – *For Hitchin representations, the first variation at the Fuchsian locus of the largest eigenvalue λ_γ of the holonomy along a simple closed geodesic γ of hyperbolic length ℓ_γ along a standard Hitchin deformation given by $q \in H^0(X, K^k)$, is*

$$\mathrm{d}\log\lambda_\gamma(\psi^0(q)) = \frac{(-1)^k(n-1)!}{2^{k-2}(n-k)!}\int_0^{\ell_\gamma} \Re\left(q(\gamma,\ldots,\gamma)\right)\,\mathrm{d}t.$$

The complete result, Theorem 4.0.1, also gives the variation of the other eigenvalues, and Corollary 4.0.3 gives the variation of the trace. The proof of Gardiner's original formula makes use of the theory of quasiconformal maps. For Hitchin representations, no such technique is available, and our proof is purely gauge theoretic. Finally, by a formula in Hejhal [20] (attributed to Petersson) the right hand side of the equation above can be interpreted as the L^2-pairing of q with the *relative Poincaré series* $\Theta_\gamma^{(k)}$ associated to γ (cf. Section 2.1.3 and Proposition 2.1.1).

The second component in the proof of Theorem 1.0.1 is a relationship, proved in Section 6, between the *variance* and the L^2-metrics for holomorphic differentials. The correspondence with the variance metric in the case of quadratic differentials has been discussed using a different framework—but belonging to the same circle of ideas—in McMullen [29]. Observe that the computation of the actual coefficients in Theorem 1.0.1 requires some technical and careful computations. However, the fact that the two metrics are proportional is a relatively easier result that is obtained earlier in the proof once the main background has been settled.