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HEAT KERNEL ASYMPTOTICS, LOCAL INDEX THEOREM AND TRACE INTEGRALS FOR CAUCHY-RIEMANN MANIFOLDS WITH S^1 ACTION

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Jih-Hsin CHENG, Chin-Yu HSIAO
& I-Hsun TSAI

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Diffusion

Maison de la SMF
Case 916 - Luminy
13288 Marseille Cedex 9
France
`commandes@smf.emath.fr`

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`memoires@smf.emath.fr` • `http://smf.emath.fr/`

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Jih-Hsin Cheng

Chin-Yu Hsiao

I-Hsun Tsai

Société Mathématique de France 2019

Jih-Hsin Cheng

Institute of Mathematics, Academia Sinica and National Center for Theoretical Sciences, 6F, Astronomy-Mathematics Building, No.1, Sec.4, Roosevelt Road, Taipei 10617, Taiwan.

E-mail : cheng@math.sinica.edu.tw

Chin-Yu Hsiao

Institute of Mathematics, Academia Sinica, 6F, Astronomy-Mathematics Building, No.1, Sec.4, Roosevelt Road, Taipei 10617, Taiwan.

E-mail : chsiao@math.sinica.edu.tw or chinyu.hsiao@gmail.com

I-Hsun Tsai

Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan.

E-mail : ihtsai@math.ntu.edu.tw

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Abstract. – Among the transversally elliptic operators initiated by Atiyah and Singer, Kohn's $\Box_b$ operator on CR manifolds with S^1 action is a natural one of geometric significance for complex analysts. Our first main result establishes an asymptotic expansion for the heat kernel of such an operator with values in its Fourier components, which involves a contribution in terms of a distance function from lower dimensional strata of the S^1-action. Our second main result computes a local index density, in terms of *tangential* characteristic forms, on such manifolds including *Sasakian manifolds* of interest in String Theory, by showing that certain non-trivial contributions from strata in the heat kernel expansion will eventually cancel out by applying Getzler's rescaling technique to off-diagonal estimates. This leads to a local result which can be thought of as a type of local index theorem on these CR manifolds. We give examples of these CR manifolds, some of which arise from Brieskorn manifolds. Moreover in some cases, we can reinterpret Kawasaki's Hirzebruch-Riemann-Roch formula for a complex orbifold equipped with an orbifold holomorphic line bundle, as an index theorem obtained by a single integral over a smooth CR manifold. We achieve this without use of equivariant cohomology method and our method can naturally drop the contributions arising from lower dimensional strata as done in previous works.

Résumé **(Étude asymptotique du noyau de la chaleur, indice local et traces sur les variétés de Cauchy-Riemann avec action d'un cercle)**

Le laplacien de Kohn sur une variété de Cauchy-Riemann (CR) avec action transverse d'un cercle est un exemple important pour l'analyse complexe d'un opérateur transversalement elliptique. Nous établissons ici un développement asymptotique du noyau de la chaleur de ses coefficients de Fourier, qui inclut une contribution des strates singulières de l'action du cercle. Nous calculons ensuite une densité d'indice locale pour ces opérateurs en montrant, à l'aide de techniques dues à Getzler, que certaines contributions des strates singulières non-triviales dans le développement du noyau de la chaleur s'annulent ici. Ce résultat, que l'on peut interpréter comme un théorème d'indice local sur ces variétés CR, s'applique notamment aux variétés de Sasaki qui sont importantes en théorie des cordes. Nous donnons également des exemples concrets de telles variétés CR, issues notamment des variétés de Brieskorn. De plus, nous pouvons réinterpréter dans certains cas la version du théorème de Hirzebruch-Riemann-Roch pour un orbifold complexe muni d'un fibré orbifold en droites complexes due à Kawasaki comme une formule d'indice. Notre méthode évite le recours à la cohomologie équivariante et les annulations des termes issus des strates singulières surviennent naturellement.

CONTENTS

CHAPTER 1

INTRODUCTION AND STATEMENT OF THE RESULTS

1.1. Introduction and Motivation

Let $(X, T^{1,0}X)$ be a compact (with no boundary) CR manifold of dimension $2n+1$ and let $\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. For a smooth function u, we say u is CR if $\overline{\partial}_b u = 0$. In CR geometry, it is crucial to be able to produce many CR functions. When X is strongly pseudoconvex and the dimension of X is greater than or equal to five, it is well-known that the space of global smooth CR functions $H^0_b(X)$ is infinite dimensional. When X is weakly pseudoconvex or the Levi form of X has negative eigenvalues, the space of global CR functions could be trivial. In general, it is very difficult to determine when the space $H^0_b(X)$ is large.

A clue to the above phenomenon arises from the following. By $\overline{\partial}_b^2 = 0$, one has the $\overline{\partial}_b$-complex: $\cdots \to \Omega^{0,q-1}(X) \to \Omega^{0,q}(X) \to \Omega^{0,q+1}(X) \to \cdots$ and the Kohn-Rossi cohomology group: $H^q_b(X) := \frac{\operatorname{Ker} \overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)}{\operatorname{Im} \overline{\partial}_b : \Omega^{0,q-1}(X) \to \Omega^{0,q}(X)}$. As in complex geometry, to understand the space $H^0_b(X)$, one could try to establish, in the CR case, a Hirzebruch-Riemann-Roch theorem for $\sum_{j=0}^{n} (-1)^j \dim H^j_b(X)$, an analog of the Euler characteristic, and to prove vanishing theorems for $H^j_b(X)$, $j \geqslant 1$. The first difficulty with such an approach lies in the fact that $\dim H^j_b(X)$ could be infinite for some j. Let's say more about this in the following.

If X is strongly pseudoconvex and of dimension $\geqslant 5$, it is well-known that $\overline{\partial}_b : \Omega^{0,0}(X) \to \Omega^{0,1}(X)$ is not hypoelliptic and for $q \geqslant 1$, $q \neq n$, $\Box^{(q)}_b : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$ is hypoelliptic so that $\dim H^0_b(X) = \infty$ and $\dim H^q_b(X) < \infty$ for $q \geqslant 1$, $q \neq n$. In other cases if the Levi form of X has exactly one negative, $n-1$ positive eigenvalues on X and $n > 3$, it is well-known that $\dim H^1_b(X) = \infty$, $\dim H^{n-1}_b(X) = \infty$ and $\dim H^q_b(X) < \infty$, for $q \notin \{1, n-1\}$. With these possibly infinite dimensional spaces, we have the trouble with defining the Euler characteristic $\sum_{j=0}^{n} (-1)^j \dim H^j_b(X)$ properly.

Another line of thought lies in the fact that the space $H^q_b(X)$ is related to the Kohn Laplacian $\Box^{(q)}_b = \overline{\partial}^*_b\,\overline{\partial}_b + \overline{\partial}_b\,\overline{\partial}^*_b : \Omega^{0,q}(X) \to \Omega^{0,q}(X)$. One can try to define the associated heat operator $e^{-t\Box^{(q)}_b}$ by using spectral theory and then the small t behavior of $e^{-t\Box^{(q)}_b}$ is closely related to the dimension of $H^q_b(X)$. Unfortunately without any Levi curvature assumption, $\Box^{(q)}_b$ is not hypoelliptic in general and it is unclear how to determine the small t behavior of $e^{-t\Box^{(q)}_b}$. Even if $\Box^{(q)}_b$ is hypoelliptic, it is still difficult to calculate the local density.

We are led to ask the following question.

Question 1.1. – *Can we establish some kind of heat kernel asymptotic expansions for Kohn Laplacian and obtain a CR Hirzebruch-Riemann-Roch theorem (not necessarily the usual ones) on some class of CR manifolds?*

It turns out that the class of CR manifolds with S^1 action is a natural choice for the above question. On this class of CR manifolds, the geometrical significance of Kohn's $\Box_b$ operator is connected with transversally elliptic operators initiated by Atiyah and Singer [**2**] (see Folland and Kohn [**35**, p.113]). Three dimensional (strongly pseudoconvex) CR manifolds with S^1 action have been intensively studied back to 1990s in relation to the CR embeddability problem. We refer the reader to the works [**26**] and [**49**] of Charles Epstein and Laszlo Lempert respectively. (see more comments on this in Section 1.3). Another related paper is about CR structure on Seifert manifolds by Kamishima and Tsuboi [**45**] (cf. Remark 1.16). In our present paper the CR manifold with S^1 action is not restricted to the three dimensional case.

To motivate our approach, let's first look at the case which can be reduced to complex geometry. Consider a compact complex manifold M of dimension n and let $(L, h^L) \to M$ be a holomorpic line bundle over M, where h^L denotes a Hermitian fiber metric of L. Let $(L^*, h^{L^*}) \to M$ be the dual bundle of (L, h^L) and put $X = \left\{v \in L^*;\ |v|^2_{h^{L^*}} = 1\right\}$. We call X the circle bundle of (L^*, h^{L^*}). It is clear that X is a compact CR manifold of dimension $2n+1$. Clearly X is equipped with a natural (globally free) S^1 action (by acting on the circular fiber). Let $T \in C^\infty(X, TX)$ be the real vector field induced by the S^1 action, that is, $Tu = \frac{\partial}{\partial\theta}(u(e^{-i\theta}\circ x))|_{\theta=0}$, $u \in C^\infty(X)$. This S^1 action is *CR and transversal*, i.e., $[T, C^\infty(X, T^{1,0}X)] \subset C^\infty(X, T^{1,0}X)$ and $\mathbb{C}T(x) \oplus T^{1,0}_xX \oplus T^{0,1}_xX = \mathbb{C}T_xX$ respectively. For each $m \in \mathbb{Z}$ and $q = 0, 1, 2, \ldots, n$, put

$$\begin{aligned}\Omega^{0,q}_m(X) :&= \left\{u \in \Omega^{0,q}(X);\ Tu = -imu\right\}\\ &= \left\{u \in \Omega^{0,q}(X);\ u(e^{-i\theta}\circ x) = e^{-im\theta}u(x), \forall\theta \in [0, 2\pi[\right\}.\end{aligned}$$

Since $\overline{\partial}_bT = T\overline{\partial}_b$, we have $\overline{\partial}_{b,m} = \overline{\partial}_b : \Omega^{0,q}_m(X) \to \Omega^{0,q+1}_m(X)$. We consider the cohomology group: $H^q_{b,m}(X) := \frac{\operatorname{Ker}\overline{\partial}_{b,m}:\Omega^{0,q}_m(X)\to\Omega^{0,q+1}_m(X)}{\operatorname{Im}\overline{\partial}_{b,m}:\Omega^{0,q-1}_m(X)\to\Omega^{0,q}_m(X)}$, and call it the m-th S^1 Fourier component of the Kohn-Rossi cohomology group.

The following result can be viewed as the starting point of this paper. Note $\Omega^{0,q}(M, L^m)$ denotes the space of smooth sections of $(0, q)$ forms on M with values in L^m (m-th power of L) and $H^q(M, L^m)$ the q-th $\overline{\partial}$-Dolbeault cohomology group with values in L^m.

THEOREM 1.2. – *For every $q = 0, 1, 2, \ldots, n$, and every $m \in \mathbb{Z}$, there is a bijective map $A_m^{(q)} : \Omega_m^{0,q}(X) \to \Omega^{0,q}(M, L^m)$ such that $A_m^{(q+1)}\overline{\partial}_{b,m} = \overline{\partial} A_m^{(q)}$ on $\Omega_m^{0,q}(X)$. Hence, $\Omega_m^{0,q}(X) \cong \Omega^{0,q}(M, L^m)$ and $H^q_{b,m}(X) \cong H^q(M, L^m)$. In particular $\dim H^q_{b,m}(X) < \infty$ and $\sum_{j=0}^{n}(-1)^j \dim H^j_{b,m}(X) = \sum_{j=0}^{n}(-1)^j \dim H^j(M, L^m)$.*

Theorem 1.2 is probably known to the experts. As a precise reference is not easily available (see, however, Folland and Kohn [**35**] p.113), we will give a proof of Theorem 1.2 in Section 1.5 for the convenience of the reader.

In this paper by *Kodaira Laplacian* we mean the Laplacian $\square_m^{(q)}$ on L^m-valued $(0, q)$ forms (on M) associated with the $\overline{\partial}$ operator, a term we borrow from the work of Ma and Marinescu [**50**]. Let $e^{-t\square_m^{(q)}}$ be the associated heat operator. It is well-known that $e^{-t\square_m^{(q)}}$ admits an asymptotic expansion as $t \to 0^+$. Consider $B_m(t) := (A_m^{(q)})^{-1} \circ e^{-t\square_m^{(q)}} \circ A_m^{(q)}$ ($A_m^{(q)}$ as in the theorem above). Let $\square_{b,m}^{(q)}$ be the Kohn Laplacian (on X) acting on (the m-th S^1 Fourier component of) $(0, q)$ forms, with $e^{-t\square_{b,m}^{(q)}}$ the associated heat operator.

A word of caution is in order. We made no use of metrics for stating Theorem 1.2. However, to define those Laplacians above an appropriate choice of metrics is needed (for *adjoint* of an operator) so that $A_m^{(q)}$ of Theorem 1.2 also preserves the chosen metrics. With this set up it is fundamental that (cf. Proposition 5.1)

$$e^{-t\square_{b,m}^{(q)}} = ((A_m^{(q)})^{-1} \circ e^{-t\square_m^{(q)}} \circ A_m^{(q)}) \circ Q_m = B_m(t) \circ Q_m = Q_m \circ B_m(t) \circ Q_m, \tag{1.1}$$

where $Q_m : \Omega^{0,q}(X) \to \Omega_m^{0,q}(X)$ is the orthogonal projection. Hence the asymptotic expansion of $e^{-t\square_m^{(q)}}$ and (1.1) lead to an asymptotic expansion

$$e^{-t\square_{b,m}^{(q)}}(x, x) \sim t^{-n} a_n^{(q)}(x) + t^{-n+1} a_{n-1}^{(q)}(x) + \cdots . \tag{1.2}$$

One goal of this work is to establish a formula similar to (1.2) (which is however not exactly of this form) on any CR manifold with S^1 action. More precisely, due to the assumption that the S^1 action is only locally free, it turns out that $e^{-t\square_{b,m}^{(q)}}(x, x)$ cannot have the standard asymptotic expansion as (1.2). Rather, our asymptotic expansion involves a contribution in terms of a *distance function* from lower dimensional strata of the S^1 action. (See (1.18) in Theorem 1.3 for details and for our first main result.) It should be emphasized that no pseudoconvexity condition is assumed.

Roughly speaking, on the regular part of X we have

$$e^{-t\square_{b,m}^{(q)}}(x, x) \sim t^{-n} a_n^{(q)}(x) + t^{-n+1} a_{n-1}^{(q)}(x) + \cdots \mod O\Big(t^{-n} e^{-\frac{\varepsilon_0 \hat{d}(x, X_{\mathrm{sing}})^2}{t}}\Big). \tag{1.3}$$

On the whole X we have, however,

$$e^{-t\Box_{b,m}^{(q)}}(x,x) \sim t^{-n}A_n^{(q)}(t,x) + t^{-n+1}A_{n-1}^{(q)}(t,x) + \cdots . \tag{1.4}$$

The difference between (1.4) and (1.3) lies in that $A_s^{(q)}(t,x)$ in (1.4) cannot be t-independent for all s and are not canonically determined (by our method) while $a_s^{(q)}(x)$ in (1.3) are t-independent for all s and are canonically determined. This t-dependence presents a great distinction between our asymptotic expansion and those in the previous literature. It appears to have a big influence on the formulation and proof of the relevant index theorems and trace integrals. See Section 7 for more comments.

In addition to the introduction of a distance function $\hat{d}$ in (1.3) our generalization has another feature, which is pertinent to the third topic of this paper, as follows. A heat kernel result for orbifolds obtained in 2008 by Dryden, Gordon, Greenwald and Webb for the case of Laplacian on functions (see (1.30) and [**23**]) and independently by Richardson ([**59**]) seems to suggest that integrating (1.3) over X is basically a power series in $t^{\frac{1}{2}}$. See (1.30) for more. To see such a possible connection, one considers X as a fiber space over X/S^1 which is then an orbifold, and presumes boldly an analogy with "(1.2) for the orbifold case". Then by the above result [**23**], integrating (1.3) over X might give an asymptotic expansion which is a power series at most in the *fractional* power $t^{\frac{1}{2}}$ of t (cf. Theorem 1.14) (while for the case where the S^1 action is globally free, such as in the circle bundle above, the asymptotic expansion is expressed in the *integral* power of t). However, our further study shows that the coefficients of t^j for j being half-integral necessarily vanish in our present case (irrespective of the local or global freeness of the S^1 action). Despite that there is no nontrivial fractional power in the t-expansion, the *corrections*/contributions associated with the stratification of the locally free S^1 action do arise nontrivially in a proper sense. Some explicit computations about these extra terms are worked out in the main result of the final section (Section 7) regarded as the third topic of this paper.

As far as the asymptotic expansion is concerned, we remark that the approach of using Kodaira Laplacian on M (downstairs) as done above is no longer applicable to the general CR case, as the contribution of a distance function on X involved in our expansion cannot be easily forseen by use of objects in the space downstairs (an orbifold in general). (However, for *trace integrals* on invariant functions, cf. Section 7, like $\sum_m e^{-t\lambda_m}$ denoted by $I(t)$ in certain Riemannian cases, $I(t)$ has been studied asymptotically with the help of the underlying/quotient manifold/orbifold, cf. [**59**, p. 2316-2317]. See also Proposition 5.1, Remark 5.4.) We must work on the entire X from scratch with the operator being only transversally elliptic (on X). (See HRR theorem below for another instance of this idea.) Furthermore, as we make no assumption on (strong) pseudoconvexity of X, this renders the techniques usually useful in this direction by previous work (e.g., [**4**]) hardly adequate in our case. Our current approach is essentially independent of the previous methods. This technicality