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SYMPLECTIC LOCAL ROOT NUMBERS, CENTRAL CRITICAL L-VALUES, AND RESTRICTION PROBLEMS IN THE REPRESENTATION THEORY OF CLASSICAL GROUPS

by

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Abstract. — In this paper, we provide a conjectural recipe for the restriction of irreducible representations of classical groups (including metaplectic groups), to certain subgroups, generalizing our earlier work on representations of orthogonal groups. Our conjectures include the cases of Bessel and Fourier-Jacobi models. In fact, it is the standard representation of the classical group, together with its orthogonal, symplectic, hermitian, or skew-hermitian form, that plays the primary role, and not the classical group alone. All of our conjectures assume the Langlands parametrization. For classical groups over local fields, the recipe involves local epsilon factors associated to the Langlands parameter and certain summands of a fixed symplectic representation of the L-group. For automorphic representations over global fields, it involves the central critical value of this symplectic L-function.

Résumé **(Nombres de racines locales symplectiques, L-valeurs critiques centrales et problèmes de restriction en théorie de représentation des groupes classiques)**

Dans cet article, nous donnons une recette conjecturale pour la restriction à certains sous-groupes des représentations irréductibles de groupes classiques. Cela inclut les groupes métaplectiques et généralise notre travail antérieur pour les groupes orthogonaux. Nos conjectures comprennent les cas des modèles de Bessel et Fourier-Jacobi. En fait le rôle principal est joué, non par le groupe seul, mais par la représentation naturelle de ce groupe classique, munie de sa forme bilinéaire-orthogonale, symplectique, hermitienne ou anti-hermitienne selon le cas. Dans toutes nos conjectures, nous admettons que la paramétrisation de Langlands est établie. Notre recette, pour les groupes classiques sur les corps locaux, fait intervenir les facteurs epsilon locaux associés au paramètre de Langlands et certains facteurs d'une représentation symplectique fixée du L-groupe. Pour les représentations automorphes sur des corps globaux, elle fait intervenir la valeur, au centre de la bande critique, de la fonction L-symplectique correspondante.

1. Introduction

It has been almost 20 years since two of us proposed a rather speculative approach to the problem of restriction of irreducible representations from SO_n to SO_{n-1}

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[24, 25]. Our predictions depended on the Langlands parametrization of irreducible representations, using L-packets and L-parameters. Since then, there has been considerable progress in the construction of local L-packets, as well as on both local and global aspects of the restriction problem. We thought it was a good time to review the precise conjectures which remain open, and to present them in a more general form, involving restriction problems for all of the classical groups.

Let k be a local field equipped with an automorphism σ with $\sigma^2 = 1$ and let k_0 be the fixed field of σ. Let V be a vector space over k with a non-degenerate sesquilinear form and let $G(V)$ be the identity component of the classical subgroup of $\mathrm{GL}(V)$ over k_0 which preserves this form. There are four distinct cases, depending on whether the space V is orthogonal, symplectic, hermitian, or skew-hermitian. In each case, for certain non-degenerate subspaces W of V, we define a subgroup H of the locally compact group $G = G(V) \times G(W)$ containing the diagonally embedded subgroup $G(W)$, and a unitary representation ν of H. The local restriction problem is to determine

$$d(\pi) = \dim_{\mathbb{C}} \mathrm{Hom}_H(\pi \otimes \overline{\nu}, \mathbb{C}),$$

where π is an irreducible complex representation of G.

The basic cases are when $\dim V - \dim W = 1$ or 0, where ν is the trivial representation or a Weil representation respectively. When $\dim V - \dim W \geq 2$, this restriction problem is also known as the existence and uniqueness of Bessel or Fourier-Jacobi models in the literature. As in **[24]** and **[25]**, our predictions involve the Langlands parametrization, in a form suggested by Vogan **[70]**, and the signs of symplectic root numbers.

We show that the Langlands parameters for irreducible representations of classical groups (and for genuine representations of the metaplectic group) are complex representations of the Weil-Deligne group of k, of specified dimension and with certain duality properties. We describe these parameters and their centralizers in detail, before using their symplectic root numbers to construct certain distinguished characters of the component group. Our local conjecture states that there is a unique representation π in each generic Vogan L-packet, such that the dimension $d(\pi)$ is equal to 1. Furthermore, this representation corresponds to a distinguished character χ of the component group. For all other representations π in the L-packet, we predict that $d(\pi)$ is equal to 0. The precise statements are contained in Conjectures 17.1 and 17.3.

Although this material is largely conjectural, we prove a number of new results in number theory and representation theory along the way:

(i) In Proposition 5.2, we give a generalization of a formula of Deligne on orthogonal root numbers to the root numbers of conjugate orthogonal representations.

(ii) We describe the L-parameters of classical groups, and unitary groups in particular, in a much simpler way than currently exists in the literature; this is contained in Theorem 8.1.

(iii) We show in Theorem 11.1 that the irreducible representations of the metaplectic group can be classified in terms of the irreducible representations of odd special orthogonal groups; this largely follows from fundamental results of Kudla-Rallis [**44**], though the statement of the theorem did not appear explicitly in [**44**].

(iv) We prove two theorems (cf. Theorems 15.1 and 16.1) that allow us to show the uniqueness of general Bessel and Fourier-Jacobi models over non-archimedean local fields. More precisely, we show that $d(\pi) \leq 1$ (cf. Corollaries 15.3, 16.2 and 16.3), reducing this to the basic cases when $\dim W^{\perp} = 0$ or 1, which were recently established by [**4**], [**64**] and [**76**]. The same theorems allow us to reduce our local conjectures to these basic cases, as shown in Theorem 19.1.

One subtle point about our local conjecture is its apparent dependence on the choice of an additive character ψ of k_0 or k/k_0. Indeed, the choice of such a character ψ is potentially used in 3 places:

(a) the Langlands-Vogan parametrization (which depends on fixing a quasi-split pure inner form G_0 of G, a Borel subgroup B_0 of G_0, and a non-degenerate character on the unipotent radical of B_0);
(b) the definition of the distinguished character χ of the component group;
(c) the representation ν of H in the restriction problem.

Typically, two of the above depend on the choice of ψ, whereas the third one doesn't. More precisely, we have:

— in the orthogonal case, none of (a), (b) or (c) above depends on ψ; this explains why this subtlety does not occur in [**24**] and [**25**].
— in the hermitian case, (a) and (b) depend on the choice of $\psi : k/k_0 \to \mathbb{S}^1$, but (c) doesn't.
— in the symplectic/metaplectic case, (a) and (c) depend on $\psi : k_0 \to \mathbb{S}^1$, but (b) doesn't.
— in the odd skew-hermitian case, (b) and (c) depend on $\psi : k_0 \to \mathbb{S}^1$, but (a) doesn't.
— in the even skew-hermitian case, (a) and (c) depend on $\psi : k_0 \to \mathbb{S}^1$ but (b) doesn't.

Given this, we check in §18 that the dependence on ψ cancels out in each case, so that our local conjecture is internally consistent with respect to changing ψ. There is, however, a variant of our local conjectures which is less sensitive to the choice of ψ, but is slightly weaker. This variant is given in Conjecture 20.1. Finally, when all the data involved are unramified, we state a more refined conjecture; this is contained in Conjecture 21.3.

After these local considerations, we study the global restriction problem, for cuspidal tempered representations of adelic groups. Here our predictions involve the central values of automorphic L-functions, associated to a distinguished symplectic representation R of the L-group. More precisely, let $G = G(V) \times G(W)$ and assume that π is an irreducible cuspidal representation of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adèles of a global field F. If the vector space $\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \bar{\nu}, \mathbb{C})$ is nonzero, our local conjecture implies that the global root number $\epsilon(\pi, R, \frac{1}{2})$ is equal to 1. If we assume π to be tempered, then our calculation of global root numbers and the general conjectures of Langlands and Arthur predict that π appears with multiplicity one in the discrete spectrum of $L^2(G(F)\backslash G(\mathbb{A}))$. We conjecture that the period integrals on the corresponding space of functions

$$f \mapsto \int_{H(k)\backslash H(\mathbb{A})} f(h) \cdot \overline{\nu(h)}\, dh$$

gives a nonzero element in $\mathrm{Hom}_{H(\mathbb{A})}(\pi \otimes \bar{\nu}, \mathbb{C})$ if and only if the central critical L-value $L(\pi, R, \frac{1}{2})$ is nonzero.

This first form of our global conjecture is given in §24, after which we examine the global restriction problem in the framework of Langlands-Arthur's conjecture on the automorphic discrete spectrum, and formulate a more refined global conjecture in §26. For this purpose, we formulate an extension of Langlands' multiplicity formula for metaplectic groups; see Conjecture 25.1.

One case in which all of these conjectures are known to be true is when $k = k_0 \times k_0$ is the split quadratic étale algebra over k_0, and V is a hermitian space over k of dimension n containing a codimension one nondegenerate subspace W. Then

$$G \cong \mathrm{GL}_n(k_0) \times \mathrm{GL}_{n-1}(k_0) \quad \text{and} \quad H \cong \mathrm{GL}_{n-1}(k_0).$$

Moreover, ν is the trivial representation. When k_0 is local, and π is a generic representation of $G = \mathrm{GL}_n(k_0) \times \mathrm{GL}_{n-1}(k_0)$, the local theory of Rankin-Selberg integrals [**34**], together with the multiplicity one theorems of [**4**], [**3**], [**66**], [**67**] and [**76**], shows that

$$\dim \mathrm{Hom}_H(\pi, \mathbb{C}) = 1.$$

This agrees with our local conjecture, as the Vogan packets for $G = \mathrm{GL}_n(k_0) \times \mathrm{GL}_{n-1}(k_0)$ are singletons. If k_0 is global and π is a cuspidal representation of $G(\mathbb{A})$, then π appears with multiplicity one in the discrete spectrum. The global theory of Rankin-Selberg integrals [**34**] implies that the period integrals over $H(k)\backslash H(\mathbb{A})$ give a nonzero linear form on π if and only if

$$L(\pi, \mathrm{std}_n \otimes \mathrm{std}_{n-1}, 1/2) \neq 0,$$

where $L(\pi, \mathrm{std}_n \otimes \mathrm{std}_{n-1}, s)$ denotes the tensor product L-function. Again, this agrees with our global conjecture, since in this case, the local and global root numbers are all equal to 1, and

$$R = \mathrm{std}_n \otimes \mathrm{std}_{n-1} + \mathrm{std}_n^\vee \otimes \mathrm{std}_{n-1}^\vee.$$

In certain cases where the global root number $\epsilon = -1$, so that the central value is zero, we also make a prediction for the first derivative in §27. The cases we treat are certain orthogonal and hermitian cases, with $\dim W^\perp = 1$. We do not know if there is an analogous conjecture for the first derivative in the symplectic or skew-hermitian cases.

In a sequel to this paper, we will present some evidence for our conjectures, for groups of small rank and for certain discrete L-packets where one can calculate the distinguished character explicitly. We should mention that in a series of amazing papers [**77, 78, 74, 75**] and [**53**], Waldspurger and Mœglin-Waldspurger have established the local conjectures for special orthogonal groups, assuming some natural properties of the characters of representations in tempered L-packets. There is no doubt that their methods will extend to the case of unitary groups.

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2. Classical groups and restriction of representations

Let k be a field, not of characteristic 2. Let σ be an involution of k having k_0 as the fixed field. If $\sigma = 1$, then $k_0 = k$. If $\sigma \neq 1$, k is a quadratic extension of k_0 and σ is the nontrivial element in the Galois group $\mathrm{Gal}(k/k_0)$.

Let V be a finite dimensional vector space over k. Let

$$\langle -, - \rangle : V \times V \to k$$

be a non-degenerate, σ-sesquilinear form on V, which is ϵ-symmetric (for $\epsilon = \pm 1$ in $k^\times$):

$$\begin{aligned} \langle \alpha v + \beta w, u \rangle &= \alpha \langle v, u \rangle + \beta \langle w, u \rangle \\ \langle u, v \rangle &= \epsilon \cdot \langle v, u \rangle^\sigma. \end{aligned}$$

Let $G(V) \subset \mathrm{GL}(V)$ be the algebraic subgroup of elements T in $\mathrm{GL}(V)$ which preserve the form $\langle -, - \rangle$:

$$\langle Tv, Tw \rangle = \langle v, w \rangle.$$

Then $G(V)$ is a classical group, defined over the field k_0. The different possibilities for $G(V)$ are given in the following table.

(k, ϵ)	$k = k_0$, $\epsilon = 1$	$k = k_0$, $\epsilon = -1$	k/k_0 quadratic, $\epsilon = \pm 1$
$G(V)$	orthogonal group $\mathrm{O}(V)$	symplectic group $\mathrm{Sp}(V)$	unitary group $\mathrm{U}(V)$

In our formulation, a classical group will always be associated to a space V, so the hermitian and skew-hermitian cases are distinct. Moreover, the group $G(V)$ is connected except in the orthogonal case. In that case, we let $\mathrm{SO}(V)$ denote the connected component, which consists of elements T of determinant $+1$, and shall refer to $\mathrm{SO}(V)$ as a connected classical group. We will only work with connected classical groups in this paper.

If one takes k to be the quadratic algebra $k_0 \times k_0$ with involution $\sigma(x, y) = (y, x)$ and V a free k-module, then a non-degenerate form $\langle -, - \rangle$ identifies the $k = k_0 \times k_0$ module V with the sum $V_0 + V_0^\vee$, where V_0 is a finite dimensional vector space over k_0 and $V_0^\vee$ is its dual. In this case $G(V)$ is isomorphic to the general linear group $\mathrm{GL}(V_0)$ over k_0.