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BOUNDARIES OF POSITIVE HOLOMORPHIC CHAINS AND THE RELATIVE HODGE QUESTION

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60^{th} birthday

Abstract. — We characterize the boundaries of positive holomorphic chains in an arbitrary complex manifold.

We then consider a compact oriented real submanifold M of dimension $2p-1$ in a compact Kähler manifold X and address the question of which relative homology classes in $H_{2p}(X, M; \mathbf{Z})$ are represented by positive holomorphic chains. Specifically, we define what it means for a class $\tau \in H_{2p}(X, M; \mathbf{Z})$ to be of type (p, p) and positive. It is then shown that τ has these properties if and only if $\tau = [T + S]$ where T is a positive holomorphic chain with $dT = \partial\tau$ and S is a positive (p, p)-current with $dS = 0$.

Résumé **(Bords de chaînes holomorphes positives et la question de Hodge relative)**

On donne une caractérisation des chaînes holomorphes positives dans une variété complexe générale.

On considère une sous-variété compacte orientée réelle M de dimension $2p-1$ dans une variété X compacte kählerienne, et on étudie les classes d'homologie relative $H_{2p}(X, M; \mathbf{Z})$ qui sont représentables par une chaîne holomorphe positive. On décrit les classes $\tau \in H_{2p}(X, M; \mathbf{Z})$ de type (p, p) positives. On montre que τ possède cette propriété si et seulement si $\tau = [T + S]$ où T est une chaîne holomorphe telle que $dT = \partial\tau$ et S est un courant (p, p) positif tel que $dS = 0$.

1. Introduction

In the first part of this note we establish a general result concerning boundaries of positive holomorphic chains in a complex manifold X. In the second part we address the "Relative Hodge Question": *When is a homology class $\tau \in H_{2p}(X, M; \mathbf{Z})$ represented by a positive holomorphic chain?* Assuming M is a real $(2p-1)$-dimensional submanifold we are able to give a surprisingly full answer.

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We begin our discussion of the first part by presenting some interesting special cases which are quite different in nature. The first main theorem is then formulated and proved in Section 2.

To start, suppose X compact and let Γ be a current of dimension $2p-1$ in X. By a positive holomorphic p-chain with boundary Γ we mean a finite sum $V = \sum_k m_k V_k$ with $m_k \in \mathbf{Z}^+$ and V_k an irreducible complex analytic variety of dimension p and finite volume in $X - \mathrm{supp}\Gamma$, such that $dV = \Gamma$ as currents on X.

Equip X with a hermitian metric and let ω denote its associated $(1,1)$-form. A real $(2p-1)$-form α will be called a (p,p)-*positive linking form* if

$$d^{p,p}\alpha + \frac{1}{p!}\omega^p \geq 0 \qquad \text{(strongly positive)}$$

where $d^{p,p}\alpha$ denotes the (p,p)-component of $d\alpha$. (See **[11]** or **[10]** for the definition of strongly and weakly positive currents.) The numbers $\int_\Gamma \alpha$ with α as above, will be called the (p,p)-*linking numbers* of Γ.

Theorem 1.1. — *Let $\Gamma = \sum_{k=1}^N n_k\Gamma_k$ be an integer linear combination of compact, mutually disjoint, C^1-submanifolds of dimension $2p-1$ in X, each of which has a real analytic point. Then $\Gamma = dV$ where V is a positive holomorphic p-chain if and only if the (p,p)-linking numbers of Γ are bounded below.*

Note 1.2. — The condition that the linking numbers of Γ are bounded below is easily seen to be independent of the choice of hermitian metric on X. However, for any given metric we have the precise statement that Γ bounds a positive holomorphic chain of mass $\leq \Lambda$ if and only if

$$\int_\Gamma \alpha \geq -\Lambda \qquad \text{for all } (p,p)\text{-positive linking forms } \alpha \tag{1.1}$$

Note 1.3. — We shall actually prove the theorem in the more general situation where Γ is allowed to have a “scar" set and the real analyticity assumption is replaced by a weaker “push-out” hypothesis (see Section 2). When $p > 1$, this hypothesis is satisfied at any point where the boundary is smooth and its Levi form has at least one negative eigenvalue. In all these cases, one has regularity at almost all points of Γ. This boundary regularity is discussed in **[12]** and **[10]**.

Remark 1.4. — When X is a projective surface and $p = 1$, a much stronger result is conjectured: namely, Γ bounds a positive holomorphic 1-chain if and only if

$$\int_\Gamma d^c u \geq -\Lambda \qquad \text{for all } u \in C^\infty(X) \text{ with } dd^c u + \omega \geq 0. \tag{1.2}$$

Functions u with $dd^c u + \omega \geq 0$ are called *quasi-plurisubharmonic*. They were introduced by Demailly and play an important role in complex analysis **[2]**, **[7]**. Condition (1.2) is equivalent to the condition that

$$\frac{1}{\ell}\mathrm{Link}_{\mathbf{P}}(\Gamma, Z) \geq -\Lambda \qquad \text{for all positive divisors } Z \text{ in } X - \Gamma$$

of sections $\sigma \in H^0(X, \mathcal{O}(\ell))$, $\ell > 0$, where $\mathrm{Link}_{\mathbf{P}}$ denotes the *projective linking number* introduced in [**17**]. In this form the conjecture extends to all dimensions and codimensions (for X projective) and is a consequence of the above case: $p = 1$ in surfaces. All this is established in [**16, 17**] where the conjectures are also related to the projective hull introduced in [**15**].

Although the hypothesis of Theorem 1.1 is conjecturally too strong for projective manifolds, it does give the "correct" result in the general case. For example, if X is a non-algebraic K3-surface, there appears to be no simpler condition characterizing the boundaries of positive holomorphic 1-chains.

Quite different characterizations of the boundaries of (not necessarily positive) holomorphic chains in projective manifolds appear in [**3**], [**4, 5**] and [**14**].

Remark 1.5. — The Linking Condition (1.1) forces the components of Γ to be maximally complex CR-manifolds. Maximal complexity is equivalent to the assertion that $\Gamma = \Gamma_{p-1,p} + \Gamma_{p,p-1}$ where $\Gamma_{r,s}$ denotes the Dolbeault component of Γ in bidimension (r, s). To see that this must hold, note that any $\alpha \in \mathcal{E}^{r,2p-1-r}(X)$ with $r \neq p-1, p$ satisfies $d^{p,p}\alpha + \omega \geq 0$ since $d^{p,p}\alpha = 0$.

Theorem 1.1 extends to characterize boundaries of compactly supported holomorphic chains in certain non-compact spaces. A complex n-manifold X is called *q-convex* if there exists a proper exhaustion function $f : X \to \mathbf{R}^+$ such that $dd^c f$ has at least $n - q + 1$ strictly positive eigenvalues outside some compact subset of X.

Theorem 1.6. — *Theorem 1.1 remains valid (for compactly supported holomorphic chains V) in any q-convex hermitian manifold with $q \leq p$.*

Remark 1.7. — If X is 1-convex (i.e., strongly pseudoconvex), then Theorem 1.1 is valid for all p. If, further, X admits a proper exhaustion which is strictly plurisubharmonic everywhere (i.e., X is Stein), much stronger results are known. Condition (1.1) implies maximal complexity, and for $p > 1$ this condition alone implies that Γ bounds a holomorphic p-chain [**12**]. Condition (1.1) also implies the *moment condition*: $\Gamma(\alpha) = 0$ for all $(p, p-1)$-forms α with $\overline{\partial}\alpha = 0$. When $p = 1$ this implies that Γ bounds a holomorphic 1-chain [**12**]. Results of this kind go back to Wermer [**21**].

Analogous remarks apply to results of [**13**] in the q-convex spaces $\mathbf{P}^n - \mathbf{P}^{n-q}$.

Remark 1.8. — Condition (1.1) implies that $\int_\Gamma \alpha \geq 0$ for all α with $d^{p,p}\alpha \geq 0$. If X is a Stein manifold embedded in some $\mathbf{C}^n$, this in turn implies that the linking number $\mathrm{Link}(\Gamma, Z) \geq 0$ for all algebraic subvarieties Z of codimension p in $\mathbf{C}^n - \Gamma$. By Alexander-Wermer [**1**], [**22**] this last condition alone implies that Γ bounds a positive holomorphic p-chain in X.

Theorem 1.1 also holds "locally", that is, it extends to any non-compact hermitian manifold X where neither Γ nor V are assumed to have compact support.

Theorem 1.9. — *Suppose X is a non-compact hermitian manifold, and let $\Gamma = \sum_j n_j\Gamma_j$ be a locally finite integral combination of disjointly embedded C^1-submanifolds of dimension $2p-1$, each of which has a real analytic point. Then Γ is the boundary of a holomorphic p-chain V of mass $M(V) \le \Lambda$ (whose support is a closed but not necessarily compact analytic subvariety of $X - \mathrm{supp}\Gamma$) if and only if $\int_\Gamma \alpha \ge -\Lambda$ for all (p,p)-positive linking forms α with compact support on X.*

In the last section of this paper we further weaken our hypotheses on Γ to an assumption that each component Γ_k be *residual* at some point. (See § 3 for the definition.) The concept of residual submanifolds leads to questions of some independent interest.

In Section 3 we address a question related to the Characterization Theorems above. Let $j : M \subset X$ be a compact oriented real submanifold of dimension $2p-1$ in a compact Kähler manifold X. Represent the relative homology group $H_{2p}(X, M; \mathbf{R})$ by $2p$-currents T on X with $dT = j_*S$ for some $(2p-1)$-current S on M. One can ask: *When does a given class $\tau \in H_{2p}(X, M; \mathbf{R})$ contain a positive holomorphic chain?*

As a first step we show that for every T as above and every d-closed form φ on X the pairing $T(\varphi)$ depends only on the relative class $\tau = [T]$. This allows us to introduce a real Hodge filtration on $H_{2p}(X, M; \mathbf{Z})_{\text{mod tor}}$ which extends the standard one on the subgroup $H_{2p}(X; \mathbf{Z})_{\text{mod tor}}$. It also allows us to formulate the following.

Definition 1.10. — A class $\tau \in H_{2p}(X, M; \mathbf{R})$ is a *positive (p,p)-class* if $\tau(\varphi) \ge 0$ for all $2p$-forms φ with $d\varphi = 0$ and $\varphi^{p,p} \ge 0$.

Theorem 1.11. — *Let $M \subset X$ be as above and suppose each component of M has a real analytic point. Let $\tau \in H_{2p}(X, M; \mathbf{Z})_{\text{mod tor}}$ be a positive (p,p)-class. Then there exists a positive holomorphic p-chain V on X with $dV = \partial\tau$ and a positive (p,p)-current S with $dS = 0$ such that $\tau = [V + S]$.*

In particular, if the positive classes in $H_{p,p}(X; \mathbf{Q})$ are represented by positive holomorphic chains with rational coefficients, then so are all the positive classes in $H_{p,p}(X, M; \mathbf{Q})$.

Remark 1.12. — This last result is a strengthening of the previous ones (in the Kähler case). Let τ be as in Theorem 1.11 and note that $\Gamma = \partial\tau = \sum_k n_k[M_k]$ where $M_1, ..., M_\ell$ are the connected components of M and the n_k's are integers. If τ is a positive (p,p)-class, then $\tau(d\alpha + \frac{1}{p!}\omega^p) \ge 0$ whenever $d^{p,p}\alpha + \frac{1}{p!}\omega^p \ge 0$. Therefore for any (p,p)-positive linking form α we have $\Gamma(\alpha) = (\partial\tau)(\alpha) = \tau(d\alpha) = \tau(d^{p,p}\alpha) = \tau(d^{p,p}\alpha + \frac{1}{p!}\omega^p) - \tau(\frac{1}{p!}\omega^p) \ge -\tau(\frac{1}{p!}\omega^p)$, and we conclude from Theorem 1.1 that Γ bounds a positive holomorphic p-chain V. Theorem 1.11 asserts that, moreover, the absolute class $\tau - [V]$ is represented by a positive (p,p)-current.

2. The Characterization Theorem

In this section we prove a general theorem which implies all of the results discussed in § 1 except Theorem 1.11. We shall assume throughout that X is a hermitian manifold which is not necessarily compact.

Definition 2.1. — Suppose there exists a closed subset Σ_Γ of Hausdorff $(2p-1)$-measure zero and an oriented, properly embedded, $(2p-1)$-dimensional C^1 submanifold of $X - \Sigma_\Gamma$ with connected components $\Gamma_1, \Gamma_2, \ldots$. If for given integers $n_1, n_2, \ldots$,

$$\Gamma = \sum_{k=1}^{\infty} n_k \Gamma_k$$

defines a current of locally finite mass in X which is d-closed, then Γ will be called a *scarred* $(2p-1)$*–cycle of class* C^1 in X. By a unique choice of orientation on Γ_k we assume each $n_k > 0$.

Example 2.2. — Any real analytic $(2p-1)$-cycle is automatically a scarred $(2p-1)$-cycle (see Federer [**6**, p. 433]).

Definition 2.3. — By a *positive holomorphic p-chain with boundary* Γ in X we mean a sum $V = \sum_k m_k V_k$ with $m_k \in \mathbf{Z}^+$ and V_k an irreducible p-dimensional complex analytic subvariety of $X - \operatorname{supp}\Gamma$ such that V has locally finite mass in X and $dV = \Gamma$ as currents.

Definition 2.4. — Suppose Γ is an embedded $(2p-1)$-dimensional oriented submanifold of a complex manifold. We say that Γ can be *pushed out at* $p \in \Gamma$ if there exists a complex p-dimensional submanifold-with-boundary $(V, -\Gamma)$ containing the point p (i.e., $\partial V = -\Gamma$ as oriented manifolds).

Our main result is the following.

Theorem 2.5. — *Let* Γ *be a scarred* $(2p-1)$*-cycle of class* C^1 *in* X *such that each component* Γ_k *can be pushed out at some point. Then* $\Gamma = dV$ *where* V *is a positive holomorphic p-chain with mass* $M(V) \leq \Lambda$ *if and only if the* (p,p)*-linking numbers of* Γ *are bounded below by* $-\Lambda$.

Remark 2.6. — We say Γ is *two sided at p* if there exists a complex p-dimensional submanifold V near p with $\Gamma \subset V$ near p. Note that if Γ is real analytic and maximally complex at p, then Γ is two-sided at p. Note also that if Γ is two-sided at p, then Γ can be pushed out at p.

The proof of Theorem 2.5 has two parts. First the linking condition is shown to be equivalent to the existence of a weakly positive current T of bidimension p,p satisfying $dT = \Gamma$. In the second part it is shown that the existence of a positive T with $dT = \Gamma$ together with the pushout hypothesis on Γ implies the existence of a positive holomorphic chain V with boundary Γ.