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DIFFRACTION OF SINGULARITIES FOR THE WAVE EQUATION ON MANIFOLDS WITH CORNERS

Richard MELROSE & András VASY & Jared WUNSCH

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Abstract. — We consider the fundamental solution to the wave equation on a manifold with corners of arbitrary codimension. If the initial pole of the solution is appropriately situated, we show that the singularities which are diffracted by the corners (i.e., loosely speaking, are not propagated along limits of transversely reflected rays) are smoother than the main singularities of the solution. More generally, we show that subject to a hypothesis of nonfocusing, diffracted wavefronts of any solution to the wave equation are smoother than the incident singularities. These results extend our previous work on edge manifolds to a situation where the fibers of the boundary fibration, obtained here by blowup of the corner in question, are themselves manifolds with corners.

Résumé **(Diffraction des singularités de l'équation d'onde sur les variétés à coins).** — Nous considérons la solution fondamentale à l'équation d'onde sur une variété à coins de codimension arbitraire. Si le pôle initial de la solution est situé arbitrairement, nous montrons que les singularités diffractées par les coins (autrement dit, intuitivement, ne sont pas propagées le long des limites de rayons réfléchis de manière transverse) sont plus lisses que les singularités principales de la solution. Plus généralement, nous montrons que sous une condition de non-focalisation, les fronts d'onde diffractés de toute solution de l'équation d'onde sont plus lisses que les singularités incidentes. Ces résultats étendent nos travaux précédents sur les variétés à bord, à une situation où les fibres de la fibration de bord, obtenue ici par un blow-up du coin en question, sont elles-mêmes des variétés à coins.

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CHAPTER 1

INTRODUCTION

1.1. The problem and its history

Let X_0 be a manifold with corners, of dimension n, i.e., a manifold locally modeled on $(\mathbb{R}^+)^{f+1} \times \mathbb{R}^{n-f-1}$, endowed with an incomplete metric, smooth and nondegenerate up to the boundary. We consider the wave equation

$$\Box u \equiv D_t^2 u - \Delta u = 0 \text{ on } M_0 = \mathbb{R} \times X_0, \tag{1.1.1}$$

where $D_t = \imath^{-1}(\partial/\partial t)$ and Δ is the nonnegative Laplace-Beltrami operator; we will impose either Dirichlet or Neumann conditions at ∂X_0. As is well known by the classic result of Duistermaat-Hörmander[(1)] (see [**4**]), the wavefront set of a solution u propagates along null-bicharacteristics in the interior. However, the behavior of singularities striking the boundary and corners of M_0 is considerably subtler.

Indeed the propagation of singularities for the wave equation on manifolds with boundary is already a rather subtle problem owing the the difficulties posed by "glancing" bicharacteristics, those which are tangent to the boundary. Chazarain [**1**] showed that singularities striking the boundary transversely simply reflect according to the usual law of geometric optics (conservation of energy and tangential momentum, hence "angle of incidence equals angle of reflection") for the reflection of bicharacteristics. This result was extended in [**21**] and [**22**] by showing that, at glancing points, singularities may only propagate along certain generalized bicharacteristics. The continuation of these curves may fail to be unique at (non-analytic) points of infinite-order tangency as shown by Taylor [**27**]. Whether all of these branches of bicharacteristics can carry singularities is still not known.

As was shown initially in several special examples (namely those amenable to separation of variables) the interaction of wavefront set with a corner gives rise to new, *diffractive* phenomena, in which a single bicharacteristic carrying a singularity into a corner produces singularities departing from the corner along a whole family of bicharacteristics. For instance, a ray carrying a singularity transversely into a codimension-two corner will in general produce singularities on the entire cone of rays reflected in

(1) This result, viewed in the context of hyperbolic equations, built on a considerable body of work prior to the introduction of the wavefront set; see especially [**9, 12**].

such a way as to conserve both energy and momentum tangent to the corner (see Figure 1) The first diffraction problem to be rigorously treated was that of the exterior

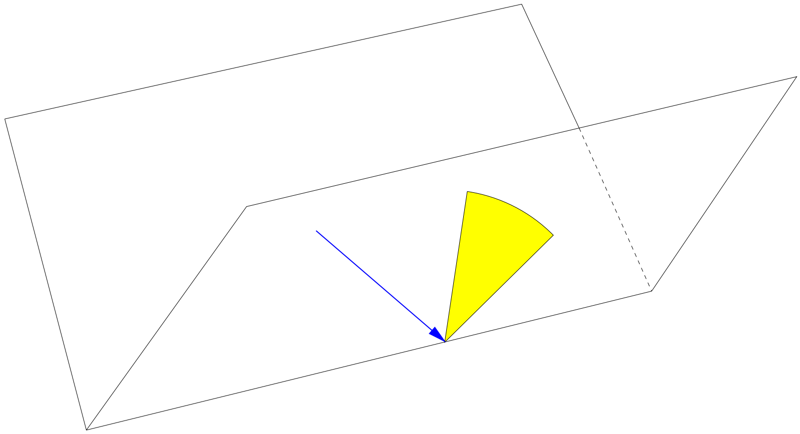

FIGURE 1. A ray carrying a singularity may strike a corner of codimension two and give rise to a whole family of diffracted singularities, conserving both energy and momentum along the corner.

of a wedge,(2) which was analyzed by Sommerfeld [**26**]; subsequently, many related examples were analyzed by Friedlander [**5**], and more generally the case of exact cones was worked out explicitly by Cheeger-Taylor [**2**], [**3**] in terms of the functional calculus for the Laplace operator on the cross section of the cone. All of these explicit examples reveal that generically a diffracted wave arises from the interaction of wavefront set of the solution with singular strata of the boundary of the manifold; this has long been understood at a heuristic level, with the geometric theory of diffraction of Keller [**8**] describing the classes of trajectories that ought to contribute to the asymptotics of the solution in various regimes.

Subsequent work has been focused primarily on characterizing the bicharacteristics on which singularities can propagate, and on describing the strength and form of the singularities that arise. The propagation of singularities on manifolds with boundary was first understood in the analytic case by Sjöstrand [**23, 24, 25**], and subsequently generalized to a very wide class of manifolds, including manifolds with corners, by Lebeau [**10, 11**]. In the $\mathscr{C}^\infty$ setting employed here, the special case of manifolds with conic singularities was studied by Melrose-Wunsch [**16**] and *edge manifolds* (i.e., cone bundles) were considered by Melrose-Vasy-Wunsch [**15**]. Vasy [**30**] obtained results analogous to Lebeau's in the case of manifolds with corners, and it is the results of this work that directly bear on the situation studied here.

While the foregoing results characterize which bicharacteristics may carry singularities for solutions to the wave equation, they ignore the question of the regularity

(2) This is not in fact a manifold with corners, but is quite closely related.