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ESSENTIAL DIMENSION

by **Alexander S. MERKURJEV**

INTRODUCTION

The essential dimension of an algebraic object is an integer that measures the complexity of the object. To motivate this notion, consider an example where the object is a quadratic extension of a field. Let F be a base field, K/F a field extension and L/K a quadratic extension. Then L is generated over K by an element α with the minimal polynomial $t^2 + at + b$ where $a, b \in K$, so L can be given by the two parameters a and b. But we can do better: if both a and b are nonzero, by scaling α, we can achieve $a = b$, i.e., just one parameter a is needed. Equivalently, we can say that the quadratic extension L/K is defined over the smaller field $K_0 = F(a)$, namely, if $L_0 = K_0[t]/(t^2+at+a)$, then $L \simeq L_0 \otimes_{K_0} K$, i.e., L/K is defined, up to isomorphism, over the field K_0 of transcendence degree at most 1 over F. On the other hand, the "generic" quadratic extension $F(t)/F(t^2)$, where t is a variable, cannot be defined over a subfield of $F(t^2)$ of transcendence degree 0. We say that the essential dimension of the class of quadratic extensions is equal to 1. Informally speaking, the essential dimension of an algebraic object is the minimal number of algebraically independent parameters one needs to define the object.

The notion of the essential dimension was introduced by J. Buhler and Z. Reichstein in [**8**] for the class of finite Galois field extensions with a given Galois group G and later in [**35**] was extended to the class of G-torsors for an arbitrary algebraic group G (see Section 2.4). Many classical algebraic objects such as simple algebras, quadratic and hermitian forms, algebras with involutions, etc., are closely related to the torsors of classical algebraic groups.

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The only property of a class of algebraic objects needed to define the essential dimension is that for every field extension K/F we have a set $\mathcal{F}(K)$ of isomorphism classes of objects, and for every field homomorphism $K \to L$ over F—a change of field map $\mathcal{F}(K) \to \mathcal{F}(L)$. In other words, $\mathcal{F}$ is a functor from the category *Fields*$_F$ of field extensions of F to the category of sets. The essential dimension for an arbitrary functor *Fields*$_F \to$ *Sets* was defined in [**4**].

One of the applications of the essential dimension is as follows. Suppose we would like to check whether a classification conjecture for the class of objects given by $\mathcal{F}$ holds. Usually, a classification conjecture assumes another functor $\mathcal{L}$ (a conjectural classification list) together with a morphism of functors $\mathcal{L} \to \mathcal{F}$, and the conjecture asserts that this morphism is surjective, i.e., the map $\mathcal{L}(K) \to \mathcal{F}(K)$ is surjective for every field extension K/F. Suppose we can compute the essential dimensions of $\mathcal{L}$ and $\mathcal{F}$, and it turns out that $\operatorname{ed}(\mathcal{L}) < \operatorname{ed}(\mathcal{F})$, i.e., the functor $\mathcal{F}$ is "more complex" than $\mathcal{L}$. This means that no morphism between $\mathcal{L}$ and $\mathcal{F}$ can be surjective and the classification conjecture fails. Thus, knowing the essential dimension allows us to predict the complexity of the structure. We have examples in quadratic form theory (Section 7.2) and in the theory of simple algebras (Corollaries 8.6 and 8.7).

Typically, the problem of computing the essential dimension of a functor splits into two problems of finding upper and lower bounds. To obtain an upper bound, one usually finds a classifying variety of the smallest possible dimension. Finding lower bounds is more complicated.

Let p be a prime integer. The essential p-dimension is the version of the essential dimension that ignores "prime to p effects". Usually, the essential p-dimension is easier to compute than the ordinary essential dimension.

If the algebraic structures given by a functor $\mathcal{F}$ are classified (parameterized), then the essential dimension of $\mathcal{F}$ can be estimated by counting the number of algebraically independent parameters. But the essential dimension can be computed in some cases where no classification theorem is available. The most impressive example is the structure given by the $\mathbf{Spin}_n$-torsors (equivalently, nondegenerate quadratic forms of dimension n with trivial discriminant and Clifford invariant). The classification theorem is available for $n \leq 14$ only, but the exact value of the essential dimension was computed for every n and this value is exponential in n.

The canonical dimension is a special case of the essential dimension (Section 3). The canonical dimension of a variety measures its compressibility. This can be studied by means of algebraic cycles and Chow motives (Section 8).

The notion of the essential dimension of a functor can be naturally extended to the categories fibered in groupoids. This allows us to unite the definitions of the essential dimension of algebraic varieties and algebraic groups. The essential dimension