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SMOOTH K-THEORY

by

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Dedicated to Jean-Michel Bismut on the occasion of his 60th birthday

Abstract. — In this paper we consider smooth extensions of cohomology theories. In particular we construct an analytic multiplicative model of smooth K-theory. We further introduce the notion of a smooth K-orientation of a proper submersion $p\colon W \to B$ and define the associated push-forward $\hat{p}_! : \hat{K}(W) \to \hat{K}(B)$. We show that the push-forward has the expected properties as functoriality, compatibility with pull-back diagrams, projection formula and a bordism formula.

We construct a multiplicative lift of the Chern character $\hat{\mathbf{ch}} : \hat{K}(B) \to \hat{H}(B, \mathbb{Q})$, where $\hat{H}(B, \mathbb{Q})$ denotes the smooth extension of rational cohomology, and we show that $\hat{\mathbf{ch}}$ induces a rational isomorphism.

If $p\colon W \to B$ is a proper submersion with a smooth K-orientation, then we define a class $A(p) \in \hat{H}^{\mathrm{ev}}(W, \mathbb{Q})$ (see Lemma 6.17) and the modified push-forward $\hat{p}_!^A := \hat{p}_!(A(p) \cup \dots) : \hat{H}(W, \mathbb{Q}) \to \hat{H}(B, \mathbb{Q})$. One of our main results lifts the cohomological version of the Atiyah-Singer index theorem to smooth cohomology. It states that $\hat{p}_!^A \circ \hat{\mathbf{ch}} = \hat{\mathbf{ch}} \circ \hat{p}_!$.

Résumé **(K-théorie différentiable).** — Dans cet article, nous considérons des extensions différentielles des théories cohomologiques. En particulier, nous construisons un modèle analytique multiplicatif de la K-théorie différentielle. Nous introduisons les K-orientations différentielles d'une submersion propre $p\colon W \to B$. Nous contruisons une application d'intégration associée: $\hat{p}_!\colon \hat{K}(W) \to \hat{K}(B)$; et nous démontrons les propriétés attendues, telles que la fonctorialité, la compatibilité aux pull-backs, des formules de projection et de bordisme.

Nous construisons un relèvement multiplicatif du caractère de Chern $\hat{\mathbf{ch}} : \hat{K}(B) \to \hat{H}(B, \mathbb{Q})$, où $\hat{H}(B, \mathbb{Q})$ est une extension différentielle de la cohomologie rationnelle, et nous démontrons que $\hat{\mathbf{ch}}$ induit un isomorphisme rationnel.

Si $p\colon W \to B$ est une submersion propre munie d'une K-orientation différentielle, nous définissons une classe $A(p) \in \hat{H}^{\mathrm{ev}}(W, \mathbb{Q})$ (compare Lemma 6.17) et une application d'intégration modifiée $\hat{p}_!^A := \hat{p}_!(A(p) \cup \dots) : \hat{H}(W, \mathbb{Q}) \to \hat{H}(B, \mathbb{Q})$. L'un de

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nos résultats principaux est une version en cohomologie différentielle du théorème d'indice d'Atiyah-Singer, pour laquelle $\hat{p}^A_! \circ \hat{\mathbf{ch}} = \hat{\mathbf{ch}} \circ \hat{p}_!$.

1. Introduction

1.1. The main results

1.1.1. — In this paper we construct a model of a smooth extension of the generalized cohomology theory K, complex K-theory. Historically, the concept of smooth extensions of a cohomology theory started with smooth integral cohomology [**24**], also called real Deligne cohomology, see [**16**]. A second, geometric model of smooth integral cohomology is given in [**24**], where the smooth integral cohomology classes were called differential characters. One important motivation of its definition was that one can associate natural differential characters to hermitean vector bundles with connection which refine the Chern classes. The differential character in degree two even classifies hermitean line bundles with connection up to isomorphism. The multiplicative structure of smooth integral cohomology also encodes cohomology operations, see [**29**].

The holomorphic counterpart of the theory became an important ingredient of arithmetic geometry.

1.1.2. — Motivated by the problem of setting up lagrangians for quantum field theories with differential form field strength it was argued in [**27**], [**26**] that one may need smooth extensions of other generalized cohomology theories. The choice of the generalized cohomology theory is here dictated by a charge quantization condition, which mathematically is reflected by a lattice in real cohomology. Let N be a graded real vector space such that the field strength lives in $\Omega_{d=0}(B)\otimes N$, the closed forms on the manifold B with coefficients in N. Let $L(B) \subset H(B,N)$ be the lattice given by the charge quantization condition on B. Then one looks for a generalized cohomology theory h and a natural transformation $c : h(B) \to H(B,N)$ such that $c(h(B)) = L(B)$. It was argued in [**27**], [**26**] that the fields of the theory should be considered as cycles for a smooth extension $\hat{h}$ of the pair (h,c). For example, if $N = \mathbb{R}$ and the charge quantization leads to $L(B) = \mathtt{im}(H(B,\mathbb{Z}) \to H(B,\mathbb{R}))$, then the relevant smooth extension could be the smooth integral cohomology theory of [**24**].

In Subsection 1.2 we will introduce the notion of a smooth extension in an axiomatic way.

1.1.3. — [**26**] proposes in particular to consider smooth extensions of complex and real versions of K-theory. In that paper it was furthermore indicated how cycle models of such smooth extensions could look like. The goal of the present paper is to carry through this program in the case of complex K-theory.

1.1.4. — In the remainder of the present subsection we describe, expanding the abstract, our main results. The main ingredient is a construction of an analytic model of smooth K-theory, also called differentiable K-theory by some authors, using cycles and relations.

1.1.5. — Our philosophy for the construction of smooth K-theory is that a vector bundle with connection or a family of Dirac operators with some additional geometry should represent a smooth K-theory class tautologically. In this way we follow the outline in [**26**]. Our class of cycles is quite big. This makes the construction of smooth K-theory classes or transformations to smooth K-theory easy, but it complicates the verification that certain cycle level constructions out of smooth K-theory are well-defined. The great advantage of our choice is that the constructions of the product and the push-forward on the level of cycles are of differential geometric nature.

More precisely we use the notion of a geometric family which was introduced in [**19**] in order to subsume all geometric data needed to define a Bismut super-connection in one notion. A cycle of the smooth K-theory $\hat{K}(B)$ of a compact manifold B is a pair $(\mathcal{E}, \rho)$ of a geometric family $\mathcal{E}$ and an element $\rho \in \Omega(B)/\mathtt{im}(d)$, see Section 2. Therefore, cycles are differential geometric objects. Secondary spectral invariants from local index theory, namely η-forms, enter the definition of the relations (see Definition 2.10). The first main result is that our construction really yields a smooth extension in the sense of Definition 1.1.

1.1.6. — Our smooth K-theory $\hat{K}(B)$ is a contravariant functor on the category of compact smooth manifolds (possibly with boundary) with values in the category of $\mathbb{Z}/2\mathbb{Z}$-graded rings. This multiplicative structure is expected since K-theory is a multiplicative generalized cohomology theory, and the Chern character is multiplicative, too. As said above, the construction of the product on the level of cycles (Definition 4.1) is of differential-geometric nature. Analysis enters the verification of well-definedness. The main result is here that our construction produces a multiplicative smooth extension in the sense of Definition 1.2.

1.1.7. — Let us consider a proper submersion $p\colon W \to B$ with closed fibres which has a topological K-orientation. Then we have a push-forward $p_!\colon K(W) \to K(B)$, and it is an important part of the theory to extend this push-forward to the smooth extension.

For this purpose one needs a smooth refinement of the notion of a K-orientation which we introduce in 3.5. We then define the associated push-forward $\hat{p}_!\colon \hat{K}(W) \to \hat{K}(B)$, again by a differential-geometric construction on the level of cycles (17). We show that the push-forward has the expected properties: functoriality, compatibility with pull-back diagrams, projection formula, bordism formula.

1.1.8. — Let $\mathbf{V} = (V, h^V, \nabla^V)$ be a hermitean vector bundle with connection. In [**24**] a smooth refinement $\hat{\mathbf{ch}}(\mathbf{V}) \in \hat{H}(B, \mathbb{Q})$ of the Chern character was constructed. In the present paper we construct a lift of the Chern character $\mathbf{ch}\colon K(B) \to H(B, \mathbb{Q})$ to a multiplicative natural transformation of smooth cohomology theories (see (30))

$$\hat{\mathbf{ch}}\colon \hat{K}(B) \to \hat{H}(B, \mathbb{Q})$$

such that $\hat{\mathbf{ch}}(\mathbf{V}) = \hat{\mathbf{ch}}([\mathcal{V}, 0])$, where $\mathcal{V}$ is the geometric family determined by $\mathbf{V}$. We prove in Proposition 6.12 that the Chern character induces a natural isomorphism of $\mathbb{Z}/2\mathbb{Z}$-graded rings $\hat{K}(B) \otimes \mathbb{Q} \xrightarrow{\sim} \hat{H}(B, \mathbb{Q})$.

1.1.9. — If $p\colon W \to B$ is a proper submersion with a smooth K-orientation, then we define a class (see Lemma 6.17) $A(p) \in \hat{H}^{\mathrm{ev}}(W, \mathbb{Q})$ and the modified push-forward

$$\hat{p}_!^A := \hat{p}_!(A(p) \cup \cdots)\colon \hat{H}(W, \mathbb{Q}) \to \hat{H}(B, \mathbb{Q}).$$

Our index theorem 6.19 lifts the characteristic class version of the Atiyah-Singer index theorem to smooth cohomology. It states that the following diagram commutes:

$$\begin{tikzcd} \hat{K}(W) \arrow[r, "\hat{\mathbf{ch}}"] \arrow[d, "\hat{p}_!"] & \hat{H}(W, \mathbb{Q}) \arrow[d, "\hat{p}_!^A"] \\ \hat{K}(B) \arrow[r, "\hat{\mathbf{ch}}"] & \hat{H}(B, \mathbb{Q}). \end{tikzcd}$$

1.1.10. — In Subsection 1.2 we present a short introduction to the theory of smooth extensions of generalized cohomology theories. In Subsection 1.3 we review in some detail the literature about variants of smooth K-theory and associated index theorems. In Section 2 we present the cycle model of smooth K-theory. The main result is the verification that our construction satisfies the axioms given below. Section 3 is devoted to the push-forward. We introduce the notion of a smooth K-orientation, and we construct the push-forward on the cycle level. The main results are that the push-forward descends to smooth K-theory, and the verification of its functorial properties. In Section 4 we discuss the ring structure in smooth K-theory and its compatibility with the push-forward. Section 5 presents a collection of natural constructions of smooth K-theory classes. In Section 6 we construct the Chern character and prove the smooth index theorem.

1.2. A short introduction to smooth cohomology theories

1.2.1. — The first example of a smooth cohomology theory appeared under the name Cheeger-Simons differential characters in [**24**]. Given a discrete subring $\mathtt{R} \subset \mathbb{R}$ we have a functor[1] $B \mapsto \hat{H}(B, \mathtt{R})$ from smooth manifolds to $\mathbb{Z}$-graded rings. It comes with natural transformations

1. $R\colon \hat{H}(B, \mathtt{R}) \to \Omega_{d=0}(B)$ (curvature)
2. $I\colon \hat{H}(B, \mathtt{R}) \to H(B, \mathtt{R})$ (forget smooth data)
3. $a\colon \Omega(B)/\mathtt{im}(d) \to \hat{H}(B, \mathtt{R})$ (action of forms).

Here $\Omega(B)$ and $\Omega_{d=0}(B)$ denote the space of smooth real differential forms and its subspace of closed forms. The map a is of degree 1. Furthermore, one has the following properties, all shown in **[24]**.

1. The following diagram commutes
$$\begin{array}{ccc} \hat{H}(B, \mathtt{R}) & \xrightarrow{I} & H(B, \mathtt{R}) \\ \downarrow R & & \downarrow \mathtt{R}\to\mathbb{R} \\ \Omega_{d=0}(B) & \xrightarrow{dR} & H(B, \mathbb{R}), \end{array}$$
where dR is the de Rham homomorphism.
2. R and I are ring homomorphisms.
3. $R \circ a = d$,
4. $a(\omega) \cup x = a(\omega \wedge R(x))$, $\forall x \in \hat{H}(B, \mathtt{R})$, $\forall \omega \in \Omega(B)/\mathtt{im}(d)$,
5. The following sequence is exact:

$$H(B, \mathtt{R}) \to \Omega(B)/\mathtt{im}(d) \xrightarrow{a} \hat{H}(B, \mathtt{R}) \xrightarrow{I} H(B, \mathtt{R}) \to 0. \tag{1}$$

1.2.2. — Cheeger-Simons differential characters are the first example of a more general structure which is described for instance in the first section of **[26]**. In view of our constructions of examples for this structure in the case of bordism theories and K-theory, and the presence of completely different pictures like **[31]** we think that an axiomatic description of smooth cohomology theories is useful.

Let N be a $\mathbb{Z}$-graded vector space over $\mathbb{R}$. We consider a generalized cohomology theory h with a natural transformation of cohomology theories $c\colon h(B) \to H(B, N)$. The natural universal example is given by $N := h^* \otimes \mathbb{R}$, where c is the canonical transformation. Let $\Omega(B, N) := \Omega(B) \otimes_{\mathbb{R}} N$. To a pair (h, c) we associate the notion of a smooth extension $\hat{h}$. Note that manifolds in the present paper may have boundaries.

Definition 1.1. — *A smooth extension of the pair (h, c) is a functor $B \to \hat{h}(B)$ from the category of compact smooth manifolds to $\mathbb{Z}$-graded groups together with natural transformations*

1. $R\colon \hat{h}(B) \to \Omega_{d=0}(B, N)$ *(curvature)*
2. $I\colon \hat{h}(B) \to h(B)$ *(forget smooth data)*

[1] In the literature, this group is sometimes denoted by $\hat{H}(B, \mathbb{R}/\mathtt{R})$, possibly with a degree-shift by one.