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THE COHOMOLOGICAL EQUATION FOR PARTIALLY HYPERBOLIC DIFFEOMORPHISMS

by

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Abstract. — We develop criteria for the existence and regularity of solutions to the cohomological equation over an accessible, partially hyperbolic diffeomorphism.

Résumé. — Nous développons des critères pour l'existence et la régularité des solutions de l'équation cohomologique au dessus d'un difféomorphisme partiellement hyperbolique et accessible.

Introduction

Let $f\colon M \to M$ be a dynamical system and let $\phi\colon M \to \mathbb{R}$ be a function. Considerable energy has been devoted to describing the set of solutions to the *cohomological equation:*

$$\phi = \Phi \circ f - \Phi, \tag{1}$$

under varying hypotheses on the dynamics of f and the regularity of ϕ. When a solution $\Phi\colon M \to \mathbb{R}$ to this equation exists, then ϕ is a called *coboundary*, for in the appropriate cohomology theory we have $\phi = d\Phi$. For historical reasons, a solution Φ to (1) is called a *transfer function.* The study of the cohomological equation has seen application in a variety of problems, among them: smoothness of invariant measures and conjugacies; mixing properties of suspended flows; rigidity of group actions; and geometric rigidity questions such as the isospectral problem. This paper studies solutions to the cohomological equation when f is a partially hyperbolic diffeomorphism and ϕ is C^r, for some real number $r > 0$.

A *partially hyperbolic diffeomorphism* $f\colon M \to M$ of a compact manifold M is one for which there exists a nontrivial, Tf-invariant splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ and a Riemannian metric on M such that vectors in E^s are uniformly contracted by Tf in this metric, vectors in E^u are uniformly expanded, and

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the expansion and contraction rates of vectors in E^c is dominated by the corresponding rates in E^u and E^s, respectively. An *Anosov diffeomorphism* is one for which the bundle E^c is trivial.

In the case where f is an Anosov diffeomorphism, there is a wealth of classical results on this subject, going back to the seminal work of Livšic, which we summarize here in Theorem 0.1. Here and in the rest of the paper, the notation $C^{k,\alpha}$, for $k \in \mathbb{Z}_+$, $\alpha \in (0,1]$, means C^k, with α-Hölder continuous kth derivative (where $C^{0,\alpha}$, $\alpha \in (0,1]$ simply means α-Hölder continuous). For $\alpha \in (0,1)$, C^α means α-Hölder continuous. More generally, if $r > 0$ is not an integer, then we will also write C^r for $C^{\lfloor r \rfloor, r - \lfloor r \rfloor}$.

Theorem 0.1. — **[23, 24, 25, 15, 16, 28, 19, 26]** *Let $f\colon M \to M$ be an Anosov diffeomorphism and let $\phi : M \to \mathbb{R}$ be Hölder continuous.*

I. Existence of solutions. *If f is C^1 and transitive, then (1) has a continuous solution Φ if and only if $\sum_{x\in\mathcal{O}} \phi(x) = 0$, for every f-periodic orbit $\mathcal{O}$.*

II. Hölder regularity of solutions. *If f is C^1, then every continuous solution to (1) is Hölder continuous.*

III. Measurable rigidity. *Let f be C^2 and volume-preserving. If there exists a measurable solution Φ to (1), then there is a continuous solution Ψ, with $\Psi = \Phi$ a.e.*

More generally, if f is C^r and topologically transitive, for $r > 1$, and μ is a Gibbs state for f with Hölder potential, then the same result holds: if there exists a measurable function Φ such that (1) holds μ-a.e., then there is a continuous solution Ψ, with $\Psi = \Phi$, μ-a.e.

IV. Higher regularity of solutions. *Suppose that $r > 1$ is not an integer, and suppose that f and ϕ are C^r. Then every continuous solution to (1) is C^r.*

If f and ϕ are C^1, then every continuous solution to (1) is C^1.

If f and ϕ are real analytic, then every continuous solution to (1) is real analytic.

There are several serious obstacles to overcome in generalizing these results to partially hyperbolic systems. For one, while a transitive Anosov diffeomorphism has a dense set of periodic orbits, a transitive partially hyperbolic diffeomorphism might have *no* periodic orbits (for an example, one can take the time-t map of a transitive Anosov flow, for an appropriate choice of t). Hence the hypothesis appearing in part I can be empty: the vanishing of $\sum_{x\in\mathcal{O}} \phi(x)$ for every periodic orbit of f cannot be a complete invariant for solving (1).

This first obstacle was addressed by Katok and Kononenko **[20]**, who defined a new obstruction to solving equation (1) when f is partially hyperbolic. To define this obstruction, we first define a relevant collection of paths in M, called su-paths, determined by a partially hyperbolic structure.

The stable and unstable bundles E^s and E^u of a partially hyperbolic diffeomorphism are tangent to foliations, which we denote by $\mathcal{W}^s$ and $\mathcal{W}^u$ respectively **[5]**. The leaves of $\mathcal{W}^s$ and $\mathcal{W}^u$ are contractible, since they are increasing unions of submanifolds diffeomorphic to Euclidean space. An *su-path* in M is a concatenation of finitely many subpaths, each of which lies entirely in a single leaf of $\mathcal{W}^s$ or a single leaf of $\mathcal{W}^u$. An *su-loop* is an *su*-path beginning and ending at the same point.

We say that a partially hyperbolic diffeomorphism $f : M \to M$ is *accessible* if any point in M can be reached from any other along an *su*-path. The *accessibility class of* $x \in M$ is the set of all $y \in M$ that can be reached from x along an *su*-path. Accessibility means that there is one accessibility class, which contains all points. Accessibility is a key hypothesis in most of the results that follow. We remark that Anosov diffeomorphisms are easily seen to be accessible, by the transversality of E^u and E^s and the connectedness of M.

Any finite tuple of points $(x_0, x_1, \ldots, x_k)$ in M with the property that x_i and x_{i+1} lie in the same leaf of either $\mathcal{W}^s$ or $\mathcal{W}^u$, for $i = 0, \ldots, k-1$, determines an *su*-path from x_0 to x_k; if in addition $x_k = x_0$, then the sequence determines an *su*-loop. Following **[1]**, we call such a tuple $(x_0, x_1, \ldots, x_k)$ an *accessible sequence* and if $x_0 = x_k$, an *accessible cycle* (the term *periodic cycle* is used in **[20]**).

For f a partially hyperbolic diffeomorphism, there is a naturally-defined *periodic cycles functional*

$$PCF\colon \{\text{accessible sequences}\} \times C^\alpha(M) \to \mathbb{R}.$$

which was introduced in **[20]** as an obstruction to solving (1). For $x \in M$ and $x' \in \mathcal{W}^u(x)$, we define:

$$PCF_{(x,x')}\phi = \sum_{i=1}^{\infty} \phi(f^{-i}(x)) - \phi(f^{-i}(x')),$$

and for $x' \in \mathcal{W}^s(x)$, we define:

$$PCF_{(x,x')}\phi = \sum_{i=0}^{\infty} \phi(f^{i}(x')) - \phi(f^{i}(x)).$$

The convergence of these series follows from the Hölder continuity of ϕ and the expansion/contraction properties of the bundles E^u and E^s. This definition then extends to accessible sequences by setting $PCF_{(x_0,\ldots,x_k)}\phi = \sum_{i=0}^{k-1} PCF_{(x_i,x_{i+1})}(\phi)$.

Assuming a hypothesis on f called *local accessibility*(1), **[20]** proved that the closely related *relative cohomological equation:*

$$\phi = \Phi \circ f - \Phi + c, \tag{2}$$

has a solution $\Phi\colon M \to \mathbb{R}$ and $c \in \mathbb{R}$, with Φ continuous, if and only if $PCF_\gamma(\phi) = 0$, for every accessible cycle γ.

The local accessibility hypothesis in **[20]** has been verified only for very special classes of partially hyperbolic systems, and it is not known whether there exist

(1) A partially hyperbolic diffeomorphism $f\colon M \to M$ is *locally accessible* if for every compact subset $M_1 \subset M$ there exists $k \geq 1$ such that for any $\varepsilon > 0$, there exists $\delta > 0$ that for every $x, x' \in M$ with $x \in M_1$ and $d(x,x') < \delta$, there is an accessible sequence $(x = x_0, \ldots, x_k = x')$ from x to x' satisfying

$$d(x_i, x) \leq \varepsilon, \quad \text{and} \quad d_{\mathcal{W}^*}(x_{i+1}, x_i) < 2\varepsilon, \quad \text{for} \quad i = 0, \ldots, k-1$$

where $d_{\mathcal{W}^*}$ denotes the distance along the $\mathcal{W}^s$ or $\mathcal{W}^u$ leaf common to the two points.

C^1-open sets of locally accessible diffeomorphisms, or more generally, whether accessibility implies local accessibility (although this seems unlikely). Assuming the strong hypothesis that E^u and E^s are C^∞ bundles, [**20**] also showed that a continuous transfer function for a C^∞ coboundary is always C^∞.

The starting point of the results here, part I of Theorem A below, is the observation that the local accessibility hypothesis in [**20**] can be replaced simply by accessibility. Accessibility is known to hold for a C^1 open and dense subset of all partially hyperbolic systems [**14**], is C^r open and dense among partially hyperbolic systems with 1-dimensional center [**36, 7**], and is conjectured to hold for a C^r open and dense subset of all partially hyperbolic diffeomorphisms, for all $r \geq 1$ [**32**]. Thus, part I of Theorem A gives a robust counterpart of part I of Theorem 0.1 for partially hyperbolic diffeomorphisms.

Another of the aforementioned major obstacles to generalizing Theorem 0.1 to the partially hyperbolic setting is that the regularity results in part IV fail to hold for general partially hyperbolic systems. Veech [**37**] and Dolgopyat [**13**] both exhibited examples of partially hyperbolic diffeomorphisms (volume-preserving and ergodic) where there is a sharp drop in regularity from ϕ to a solution Φ. These examples are not accessible. Here we show in Theorem A, part IV, that assuming accessibility and a C^1-open property called *strong r-bunching* (which incidentally is satisfied by the nonaccessible examples in [**37, 13**]), there is no significant loss of regularity between ϕ and Φ.

Part III of Theorem 0.1 is the most resistant to generalization, primarily because a general notion of Gibbs state for a partially hyperbolic diffeomorphism remains poorly understood. In the conservative setting, the most general result to date concerning ergodicity of for partially hyperbolic diffeomorphisms is due to Burns and Wilkinson [**9**], who show that every C^2, volume-preserving partially hyperbolic diffeomorphism that is center-bunched and accessible is ergodic. Center bunching is a C^1-open property that roughly requires that the action of Tf on E^c be close to conformal, relative to the expansion and contraction rates in E^s and E^u (see Section 2). Adopting the same hypotheses as in [**9**], we recover here the analogue of Theorem 0.1 part III for volume-preserving partially hyperbolic diffeomorphisms.

We now state our main result.

Theorem A. — *Let $f : M \to M$ be partially hyperbolic and accessible, and let $\phi : M \to \mathbb{R}$ be Hölder continuous.*

I. Existence of solutions. *If f is C^1, then* (2) *has a continuous solution Φ for some $c \in \mathbb{R}$ if and only if $PCF_{\mathscr{C}}(\phi) = 0$, for every accessible cycle $\mathscr{C}$.*

II. Hölder regularity of solutions. *If f is C^1, then every continuous solution to* (2) *is Hölder continuous.*

III. Measurable rigidity. *Let f be C^2, center bunched, and volume-preserving. If there exists a measurable solution Φ to* (2)*, then there is a continuous solution Ψ, with $\Psi = \Phi$ a.e.*

IV. Higher regularity of solutions. *Let $k \geq 2$ be an integer. Suppose that f and ϕ are both C^k and that f is strongly r-bunched, for some $r < k-1$ or $r = 1$. If Φ is a continuous solution to (2), then Φ is C^r.*

The center bunching and strong r-bunching hypotheses in parts III and IV are C^1-open conditions and are defined in Section 2. Theorem A part IV generalizes all known C^∞ Livšic regularity results for accessible partially hyperbolic diffeomorphisms. In particular, it applies to all time-t maps of Anosov flows and compact group extensions of Anosov diffeomorphisms. Accessibility is a C^1 open and C^∞ dense condition in these classes [**8, 6**]. In dimension 3, for example, the time-1 map of any mixing Anosov flow is stably accessible [**6**], unless the flow is a constant-time suspension of an Anosov diffeomorphism.

We also recover the results of [**13**] in the context of compact group extensions of volume-preserving Anosov diffeomorphisms. Finally, Theorem A also applies to all accessible, partially hyperbolic affine transformations of homogeneous manifolds. A direct corollary that encompasses these cases is:

Corollary 0.2. — *Let f be C^∞, partially hyperbolic and accessible. Assume that $Tf|_{E^c}$ is isometric in some continuous Riemannian metric. Let $\phi\colon M \to \mathbb{R}$ be C^∞. Suppose there exists a continuous function $\Phi\colon M \to \mathbb{R}$ such that*

$$\phi = \Phi \circ f - \Phi.$$

Then Φ is C^∞. If, in addition, f preserves volume, then any measurable solution Φ extends to a C^∞ solution.

For any such f, and any integer $k \geq 2$, there is a C^1 open neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^k(M)$ such that, for any accessible $g \in \mathcal{U}$, and any C^k function $\phi\colon M \to \mathbb{R}$, if

$$\phi = \Phi \circ g - \Phi,$$

has a continuous solution Φ, then Φ is C^1 and also C^r, for all $r < k-1$. If g also preserves volume, then any measurable solution extends to a C^r solution.

The vanishing of the periodic cycles obstruction in Theorem A, part I turns out to be a practical method in many contexts for determining whether (2) has a solution. On the one hand, this method has already been used by Damjanović and Katok to establish rigidity of certain partially hyperbolic abelian group actions [**12**]; in this (locally accessible, algebraic) context, checking that the PCF obstruction vanishes reduces to questions in classical algebraic K-theory (see also [**11, 21**]). On the other hand, for a given accessible partially hyperbolic system, the PCF obstruction provides an infinite codimension obstruction to solving (2), and so the generic cocycle ϕ has no solutions to (2). This latter fact follows from recent work of Avila, Santamaria and Viana on the related question of vanishing of Lyapunov exponents for linear cocycles over partially hyperbolic systems (see [**1**], Section 9).

As part of proof of Theorem A, part II, we also prove that stable and unstable foliations of any C^1 partially hyperbolic diffeomorphism are transversely Hölder continuous (Corollary 5.3). This extends to the C^1 setting the well-known fact that