339

ASTÉRISQUE

2011

SÉMINAIRE BOURBAKI
VOLUME 2009/2010
EXPOSÉS 1012-1026

(1025) *The ACC conjecture for log canonical thresholds*

Burt TOTARO

SOCIÉTÉ MATHÉMATIQUE DE FRANCE

Publié avec le concours du CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

Séminaire BOURBAKI Juin 2010
62[e] année, 2009-2010, n° 1025, p. 371 à 385

THE ACC CONJECTURE FOR LOG CANONICAL THRESHOLDS
[after de Fernex, Ein, Mustaţă, Kollár]

by **Burt TOTARO**

INTRODUCTION

The minimal model conjecture asserts that every algebraic variety X not covered by rational curves is birational to a projective variety Y which is negatively curved in the sense that the first Chern class $c_1(Y)$ has degree ≤ 0 on all curves in Y, called a minimal model. The problem is fundamental to all kinds of classification problems in algebraic geometry. The minimal model conjecture is known for all varieties of general type [**9**] and for all varieties of dimension at most 4, but the full conjecture remains wide open. (Although the minimal model conjecture should be true in any characteristic, the results mentioned work over a field of characteristic zero. For these problems, we lose nothing by working over the complex numbers. Birkar [**7**] summarizes the known results on the minimal model conjecture in dimensions up to 5.)

Shokurov showed that the minimal model conjecture would follow from the ascending chain condition for a certain invariant of singularities, the minimal log discrepancy (mld), together with a semicontinuity property for minimal log discrepancies [**24**]. The minimal log discrepancy is a rational number associated to a given singularity, with bigger numbers corresponding to milder singularities. Shokurov's ACC conjecture says that the set of minimal log discrepancies of all singularities of a given dimension is a subset of the real line that contains no infinite increasing sequence. Let us say vaguely why this would imply the minimal model conjecture. Starting from any projective variety, after contracting finitely many divisors, we can make it closer and closer to a minimal model by a sequence of birational maps called flips. We get a minimal model if we can show that no infinite sequence of flips is possible. But each flip improves the singularities as measured by minimal log discrepancies, and so the ACC conjecture would imply that the sequence of flips terminates.

Shokurov also conjectured the ACC property for another invariant of singularities, the log canonical threshold (lct). This conjecture should be a natural preliminary to the ACC conjecture for minimal log discrepancies, since log canonical thresholds are simpler than minimal log discrepancies and have been studied for a long time in singularity theory. In particular, ACC for log canonical thresholds is known in dimension at most 3 while ACC for minimal log discrepancies is known only in dimension at most 2, by Alexeev [**1, 2**]; moreover, ACC for minimal log discrepancies in dimension 3 would imply ACC for log canonical thresholds in dimension 4 [**10**, Corollary 1.10]. Log canonical thresholds have an elementary analytic definition: for an analytic function f on $\mathbf{C}^n$, the log canonical threshold of the hypersurface $f = 0$ at a point p is the supremum of the real numbers s such that $|f|^{-s}$ is L^2 near p. Estimates of log canonical thresholds have a variety of applications in algebraic geometry, including the construction of Kähler-Einstein metrics on many Fano varieties [**11**] and the proof of non-rationality for many Fano varieties [**12**].

The ACC conjecture for log canonical thresholds has some implication for the minimal model conjecture, albeit a limited one. By Birkar, the minimal model conjecture in dimension $n-1$ (for pairs (X, B) with $K_X + B$ pseudo-effective) implies the minimal model conjecture for pairs (X, B) of dimension n *with* $K_X + B$ *effective* [**8**]. If we also know ACC for log canonical thresholds on singular varieties of dimension n, then we can deduce termination of flips for pairs (X, B) of dimension n with $K_X + B$ effective [**6**]. Termination is a stronger statement than existence of minimal models (because it says that any sequence of flips will lead to a minimal model).

A recent advance is the proof of ACC for log canonical thresholds on smooth varieties of any dimension by de Fernex, Ein, and Mustaţă [**13**]. Their method covers many singular varieties as well, including quotient singularities and local complete intersections. In dimension 3, every terminal singularity is a quotient of a hypersurface singularity by a finite group, and so de Fernex-Ein-Mustaţă's methods reprove the general ACC conjecture for log canonical thresholds in dimension 3. In dimensions at least 4, there seems to be no hope of a comparably explicit description of terminal singularities. Nonetheless, de Fernex-Ein-Mustaţă's work provides striking new evidence for the ACC conjectures, suggesting that they form a plausible approach toward the minimal model conjecture. The final version of de Fernex-Ein-Mustaţă's argument, incorporating contributions by Kollár, is short and simple.

This exposition owes a lot to Kollár's excellent survey of the ACC conjecture for log canonical thresholds [**18**]. Thanks to Ofer Gabber for suggesting Corollary 1.7.

1. INTRODUCTION TO LOG CANONICAL THRESHOLDS

Definition 1.1. — *Let f be a holomorphic function in a neighborhood of a point $p \in \mathbf{C}^n$. The log canonical threshold of f at p is the number $c = \mathrm{lct}_p(f)$ such that*

— *$|f|^{-s}$ is L^2 in a neighborhood of p for $s < c$, and*
— *$|f|^{-s}$ is not L^2 in a neighborhood of p for $s > c$.*

Thus $\mathrm{lct}_p(f) = \infty$ if $f(p) \neq 0$, and by convention $\mathrm{lct}_p(0) = 0$.

The log canonical threshold is a natural measure of the complexity of the zero set of f near p. It was considered by Atiyah [**4**] and Bernstein [**5**], and further explored by Arnold, Gusein-Zade and Varchenko [**3**, v. 2, Section 13.1.5] and Kollár [**16**]. The main textbook treatment of log canonical thresholds is in Lazarsfeld's book [**21**, Section 9.3].

Shokurov had the idea that the set of all possible values of log canonical thresholds in a given dimension should have special properties [**23**].

Definition 1.2. — *Let $\mathcal{HT}_n$ be the set of log canonical thresholds of all possible holomorphic functions of n variables vanishing at 0. That is,*

$$\mathcal{HT}_n = \{\mathrm{lct}_0(f) : f \in O_{0,\mathbf{C}^n},\ f(0) = 0\} \subset \mathbf{R}.$$

The name $\mathcal{HT}_n$ indicates that these are "hypersurface thresholds". Indeed, it is easy to see that $\mathrm{lct}_0(gf) = \mathrm{lct}_0(f)$ for $g(0) \neq 0$, which says that $\mathrm{lct}_0(f)$ only depends on the hypersurface $\{f = 0\}$ near $0 \in \mathbf{C}^n$ (considered with multiplicities). We get the same set $\mathcal{HT}_n$ if we allow f to run through all polynomials or all formal power series over any algebraically closed field of characteristic zero, using the algebraic definition of the log canonical threshold in Section 3 [**18**, Section 5].

The function $|z|^{-s}$ is L^2 on a neighborhood of the origin if and only if $s < 1$. It follows that, for a holomorphic function f of one variable,

$$\mathrm{lct}_p(f) = \frac{1}{\mathrm{mult}_p(f)}.$$

As a result,

$$\mathcal{HT}_1 = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, 0\right\}.$$

There is also a complete description of the set $\mathcal{HT}_2$, by Varchenko [**26**], [**19**, Theorem 6.40], [**18**, Equation (15.5)]:

$$\mathcal{HT}_2 = \left\{\frac{c_1 + c_2}{c_1 c_2 + a_1 c_2 + a_2 c_1} : a_1, a_2, c_1, c_2 \in \mathbf{N},\ c_1 + c_2 \geq 1,\right.$$
$$\left. a_1 + c_1 \geq \max\{2, a_2\}, a_2 + c_2 \geq \max\{2, a_1\}\right\} \cup \{0\}.$$

The sets $\mathcal{HT}_n$ are not known for $n \geq 3$, and it may be unreasonable to expect an explicit description. Nonetheless, they have remarkable properties. First, Atiyah used resolution of singularities to prove:

LEMMA 1.3. — *All log canonical thresholds are rational and lie between* 0 *and* 1. *That is,* $\mathcal{HT}_n \subset \mathbf{Q} \cap [0,1]$.

We will discuss de Fernex, Ein, and Mustaţă's theorem:

THEOREM 1.4 (ACC conjecture, smooth case). — *For any n, there is no infinite increasing sequence in* $\mathcal{HT}_n$.

By contrast, every rational number between 0 and 1 is the log canonical threshold of a holomorphic function in some number of variables. Also, there are many decreasing sequences of log canonical thresholds in a given dimension, as we see from the example [**3**, v. 2, Section 13.3.5], [**16**, Proposition 8.21]:

LEMMA 1.5. — *We have*

$$\mathrm{lct}_0(z_1^{a_1} + \cdots + z_n^{a_n}) = \min\left\{1, \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right\}.$$

Kollár described all limits of decreasing sequences of log canonical thresholds on smooth varieties of dimension n:

THEOREM 1.6 (Accumulation conjecture, smooth case). — *The set of accumulation points of* $\mathcal{HT}_n$ *is* $\mathcal{HT}_{n-1} - \{1\}$.

In particular, $\mathcal{HT}_n$ is a closed subset of the unit interval, although it is contained in the rational numbers.

The ACC theorem implies that there is some $\epsilon_n > 0$ such that no log canonical threshold on a smooth n-dimensional variety is in $(1 - \epsilon_n, 1)$ (the smooth case of the Gap conjecture). The proofs are nonconstructive, and so there is no explicit lower bound for ϵ_n in general. There is a conjecture for the optimal value of ϵ_n. Consider the sequence defined by $c_{n+1} = c_1 \ldots c_n + 1$ starting with $c_1 = 2$. It is called Euclid's or Sylvester's sequence, and starts as:

$$2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots$$

The definition of c_i implies that

$$\sum \frac{1}{c_i} = 1 - \frac{1}{c_{n+1} - 1} = 1 - \frac{1}{c_1 \cdots c_n}.$$

In particular, by Lemma 1.5,

$$\mathrm{lct}_0(z_1^{c_1} + \cdots + z_n^{c_n}) = 1 - \frac{1}{c_{n+1} - 1}.$$