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KERVAIRE INVARIANT ONE
[after M. A. Hill, M. J. Hopkins, and D. C. Ravenel]

by **Haynes MILLER**

INTRODUCTION

Around the year 1960 the theory of surgery was developed as part of a program to classify manifolds of dimension greater than 4. Among the questions it addresses is this: Does every framed cobordism class contain a homotopy sphere?

Recall that a *framing* of a closed smooth manifold is an embedding into Euclidean space together with a trivialization of the normal bundle. A good example is given by the circle embedded in $\mathbb{R}^3$, with a framing t such that the framing normal vector fields have linking number ± 1 with the circle itself. A framed manifold is *null-bordant* if it is the boundary of a framed manifold-with-boundary, and two framed manifolds are *cobordant* if their difference is null-bordant. Cobordism classes of framed n-manifolds form an abelian group Ω_n^{fr}. The class of (S^1, t) in Ω_1^{fr} is written η.

Any closed manifold of the homotopy type of S^n admits a framing [**20**], and the seemingly absurdly ambitious question arises of whether every class contains a manifold of the homotopy type (and hence, by Smale, of the homeomorphism type, for $n > 4$) of a sphere. (Classes represented by some framing of the standard n-sphere form a cyclic subgroup of order given by the resolution of the Adams conjecture in the late 1960's.)

A result of Pontryagin from 1950 implied that the answer had to be "No" in general: $\eta^2 \neq 0$ in Ω_2^{fr}, while S^2 is null-bordant with any framing.

Kervaire and Milnor [**20**] showed that the answer is "Yes" unless $n = 4k+2$; but that there is an obstruction, the *Kervaire invariant*

$$\kappa : \Omega_{4k+2}^{\mathrm{fr}} \to \mathbb{Z}/2\mathbb{Z}\,,$$

which vanishes on a cobordism class if and only if the class contains a homotopy sphere. Kervaire's construction [**19**] of a PL 10-manifold with no smooth structure amounted to showing that $\kappa = 0$ on $\Omega^{\mathrm{fr}}_{10}$. Pontryagin had shown that $\kappa(\eta^2) \neq 0$, and in fact the Kervaire invariant is nontrivial on the square of any element of Hopf invariant one. In **[20]** Kervaire and Milnor speculated that these may be the only examples with $\kappa \neq 0$.

The Pontryagin-Thom construction establishes an isomorphism between Ω^{fr}_n and the stable homotopy group $\pi^s_n(S^0)$, and in 1969 Browder **[6]** gave a homotopy theoretic interpretation of the Kervaire invariant (Theorem 1.1 below) which implied that $\kappa = 0$ unless $4k+2$ is of the form $2(2^j - 1)$. Homotopy theoretic calculations **[27, 5, 3]** soon muddied the waters by providing examples in dimensions 30 and 62. Much effort in the 1970's went into understanding the role of these classes and attempting inductive constructions of them in all dimensions.

The focus of this report is the following result.

THEOREM 0.1 (Hill, Hopkins, Ravenel, 2009, **[15]**). — *The Kervaire invariant* $\kappa : \Omega^{\mathrm{fr}}_{4k+2} \to \mathbb{Z}/2$ *is trivial unless* $4k+2 = 2, 6, 14, 30, 62$*, or (possibly)* 126*.*

The case $4k+2 = 126$ remains open.

In his proof that κ is trivial in dimension 10, Kervaire availed himself of the current state of the art in homotopy theory (mainly work of Serre). Kervaire and Milnor relied on contemporaneous work of Adams. Over the intervening fifty years, further developments in homotopy theory have been brought to bear on the Kervaire invariant problem. In 1964, for example, Brown and Peterson brought spin bordism into play to show that κ is trivial in dimensions $8k+2$, $k > 0$, and Browder's work used the Adams spectral sequence. Over the past quarter century, however, essentially no further progress has been made on this problem till the present work.

Hill, Hopkins, and Ravenel (hereafter HHR) marshall three major developments in stable homotopy theory in their attack on the Kervaire invariant problem:

- The chromatic perspective based on work of Novikov and Quillen and pioneered by Landweber, Morava, Miller, Ravenel, Wilson, and many more recent workers;
- The theory of structured ring spectra, implemented by May and many others; and
- Equivariant stable homotopy theory, as developed by May and collaborators.

The specific application of equivariant stable homotopy theory was inspired by analogy with a fourth development, the motivic theory initiated by Voevodsky and Morel, and uses as a starting point the theory of "Real bordism" investigated by Landweber, Araki, Hu and Kriz. In their application of these ideas, HHR require significant extensions of the existing state of knowledge of this subject, and their paper provides an excellent account of the relevant parts of equivariant stable homotopy theory.

1. THE KERVAIRE INVARIANT

1.1. Geometry

Any compact smooth n-manifold M embeds into a Euclidean space, and in high codimension any two embeddings are isotopic. The normal bundle ν is thus well defined up to addition of a trivial bundle. A *framing* of M is a bundle isomorphism $t : \nu \to M \times \mathbb{R}^q$. The Pontryagin-Thom construction is the induced contravariant map on one-point compactifications, $S^{n+q} = \mathbb{R}^{n+q}_+ \to (M \times \mathbb{R}^q)_+ = M_+ \wedge S^q$, giving an element of $\pi_{n+q}(M_+ \wedge S^q)$.

This group becomes independent of q for $q > n$, and is termed the nth *stable homotopy group* $\pi^s_n(M_+) = \lim_{q\to\infty} \pi_{n+q}(M_+ \wedge S^q)$ of M. The Pontryagin-Thom construction provides a "stable homotopy theory fundamental class" $[M, t] \in \pi^s_n(M_+)$. Composing with the map collapsing M to a point gives an element of $\pi^s_n(S^0)$. A bordism between framed manifolds determines a homotopy between their Pontryagin-Thom collapse maps, and this construction gives an isomorphism from the framed bordism group to $\pi^s_n(S^0)$.

A framing of a manifold M of dimension $4k + 2$ determines additional structure in the cohomology of M. A cohomology class $x \in H^{2k+1}(M;\mathbb{F}_2)$ is represented by a well-defined homotopy class of maps $M_+ \to K(\mathbb{F}_2, 2k + 1)$. When we apply the stable homotopy functor we get a map $\pi^s_*(M_+) \to \pi^s_*(K(\mathbb{F}_2, 2k + 1))$. A calculation [**7**] shows that $\pi^s_{4k+2}(K(\mathbb{F}_2, 2k + 1)) = \mathbb{Z}/2$, so the class $[M, t]$ determines an element $q_t(x) \in \mathbb{Z}/2$.

This *Kervaire form* $q_t : H^{2k+1}(M;\mathbb{F}_2) \to \mathbb{Z}/2$ turns out to be a *quadratic refinement* of the intersection pairing $x \cdot y = \langle x \cup y, [M]\rangle$—that is to say,

$$q_t(x + y) = q_t(x) + q_t(y) + x \cdot y\,.$$

In the group (under direct sum) of isomorphism classes of finite dimensional $\mathbb{F}_2$-vector spaces with nondegenerate quadratic form, the Kervaire forms of cobordant framed manifolds are congruent modulo the subgroup generated by the *hyperbolic* quadratic space (H, q) with $H = \langle a, b\rangle$, $q(a) = q(b) = q(0) = 0$, $q(a + b) = 1$. This quotient is the *quadratic Witt group* of $\mathbb{F}_2$, and is of order 2. The element of $\mathbb{Z}/2$ corresponding to a quadratic space is given by the *Arf invariant*, namely the more popular of $\{0, 1\}$ as a value of q. The *Kervaire invariant* of (M, t) is then the Arf invariant of the quadratic space $(H^{2k+1}(M;\mathbb{F}_2), q_t)$. This defines a homomorphism

$$\kappa : \pi^s_{4k+2}(S^0) = \Omega^{\mathrm{fr}}_{4k+2} \to \mathbb{Z}/2\,.$$

1.2. Homotopy theory

Regarding κ as defined on $\pi^s_{4k+2}(S^0)$ invites the question: What is a homotopy-theoretic interpretation of the Kervaire invariant? This was answered by Browder in a landmark paper [**6**], in terms of the Adams spectral sequence.

Discussion of these matters is streamlined by use of the *stable homotopy category* $h\mathcal{S}$, first described in Boardman's thesis (1964). Its objects are called *spectra* and are designed to represent cohomology theories. There are many choices of underlying categories of spectra, but they all lead the same homotopy category $h\mathcal{S}$, which is additive, indeed triangulated, and symmetric monoidal (with tensor product given by the "smash product" $\wedge$). It is an analog of the derived category of a commutative ring. There is a "stabilization" functor Σ^∞ from the homotopy category of pointed CW complexes to $h\mathcal{S}$. It sends the two point space S^0 to the "sphere spectrum" $\Sigma^\infty S^0 = \mathbb{S}$ which serves as the unit for the smash product. The suspension functor Σ is given by smashing with $\Sigma^\infty S^1$. The *homotopy* of a spectrum E is $\pi_n(E) = [\Sigma^n\mathbb{S}, E]$, so that, for a space X, $\pi^s_n(X) = \pi_n(\Sigma^\infty X_+)$. Ordinary mod 2 cohomology of a space X (which we abbreviate to $H^*(X)$) is represented by the *Eilenberg-Mac Lane spectrum* $\mathbb{H}$—$H^n(X) = [\Sigma^\infty X_+, \Sigma^n\mathbb{H}]$—and, as explained by G. Whitehead, homology is obtained as $H_n(X) = [\Sigma^n\mathbb{S}, \Sigma^\infty X_+ \wedge \mathbb{H}]$. The cup product is represented by a structure map $\mathbb{H}\wedge\mathbb{H} \to \mathbb{H}$, making $\mathbb{H}$ into a "ring-spectrum." The unit map for this ring structure, $\mathbb{S} \to \mathbb{H}$, represents a generator of $\pi_0(\mathbb{H}) = H^0(\mathbb{S})$. The graded endomorphism algebra of the object $\mathbb{H}$ is the well-known *Steenrod algebra* $\mathcal{A}$ of stable operations on mod 2 cohomology.

Evaluation of homology gives a natural transformation (generalizing the *degree*)

$$d_X : \pi_t(X) = [\Sigma^t\mathbb{S}, X] \to \operatorname{Hom}_{\mathcal{A}}(H^*(X), H^*(\Sigma^t\mathbb{S}))$$

which is an isomorphism if $X = \mathbb{H}$. This leads to the *Adams spectral sequence*

$$E_2^{s,t}(X;\mathbb{H}) = \operatorname{Ext}^{s,t}_{\mathcal{A}}(H^*(X), \mathbb{F}_2) = \operatorname{Ext}^s_{\mathcal{A}}(H^*(X), H^*(\Sigma^t\mathbb{S})) \Longrightarrow \pi_{t-s}(X)^\wedge_2$$

for X a spectrum such that $\pi_n(X) = 0$ for $n \ll 0$ and $H_n(X)$ is finite dimensional for all n. It converges to the 2-adic completion of the homotopy groups of X. In particular,

$$E_2^{s,t} = \operatorname{Ext}^{s,t}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(\mathbb{S})^\wedge_2 \,.$$

To this day these Ext groups remain quite mysterious overall, but in [**1**] Adams computed them for $s \le 2$. $E_2^{0,t}$ is of course just $\mathbb{F}_2$ in $t = 0$. The edge homomorphism

$$e : \pi_{t-1}(\mathbb{S}) \to E_2^{1,t}$$

is the "mod 2 Hopf invariant." One interpretation of this invariant is that $e(\alpha) \neq 0$ if and only if the mapping cone $S^0 \cup_\alpha e^t$ supports a nonzero Steenrod operation of positive degree. $E_2^{1,*}$ is the dual of the module of indecomposables in the Steenrod