352

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Quantum gravity and the KPZ formula

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QUANTUM GRAVITY AND THE KPZ FORMULA [after Duplantier-Sheffield]

by **Christophe GARBAN**

1. INTRODUCTION

The study of statistical physics models in two dimensions ($d = 2$) at their *critical point* is in general a significantly hard problem (not to mention the $d = 3$ case). In the eighties, three physicists, Knizhnik, Polyakov and Zamolodchikov (KPZ) came up in **[14]** with a novel and far-reaching approach in order to understand the critical behavior of these models. Among these, one finds for example random walks, percolation as well as the Ising model. The main underlying idea of their approach is to study these models along a two-step procedure as follows:

- First of all, instead of considering the model on some regular lattice of the plane (such as $\mathbb{Z}^2$ for example), one defines it instead on a well-chosen "random planar lattice". Doing so corresponds to studying the model in its *quantum gravity* form. In the case of percolation, the appropriate choice of random lattice matches with the so-called *planar maps* which are currently the subject of an intense activity (see for example **[22]**).
- Then it remains to get back to the actual *Euclidean* setup. This is done thanks to the celebrated *KPZ formula* which gives a very precise correspondence between the geometric properties of models in their quantum gravity formulation and their analogs in the Euclidean case.

It is fair to say that the nature and the origin of such a powerful correspondence remained rather mysterious for a long time. In fact, the KPZ formula is still not rigorously established and remains a conjectural correspondence. The purpose of this survey is to explain how the recent work of Duplantier and Sheffield enables to explain some of the mystery hidden behind this KPZ formula. To summarize their

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contribution in one sentence, their work implies a beautiful interpretation of the KPZ correspondence through a uniformization of the random lattice, seen as a Riemann surface.

To fully appreciate the results by Duplantier-Sheffield, we will need to introduce beforehand several related concepts and objects. More precisely, the rest of this introduction is divided as follows: first we give a short and informal discussion about *quantum gravity*, then we introduce two universality classes of random lattices. Then we will come to the KPZ formula through a specific example (boundary of Random Walks hulls), and finally after stating the main Theorem by Duplantier-Sheffield, we will state a beautiful conjecture they made.

1.1. A first glance into quantum gravity

Quantum gravity is intimately concerned with the following naive question:

QUESTION 1.1. — *How does a "uniformly distributed" random metric on the sphere $\mathbb{S}^2$ typically look ?*

What is naive in this question is the fact that one would first need to specify what we mean by "a uniform probability measure" on the space of metrics on $\mathbb{S}^2$. It turns out that defining a natural model of random metric on the sphere $\mathbb{S}^2$ already is a difficult and interesting problem. To illustrate this, let us ask a similar naive question in a one-dimensional setting:

QUESTION 1.2. — *For any $a, b \in \mathbb{R}^d$, how does a "uniformly distributed" path $\gamma : [0,1] \to \mathbb{R}^d$ going from a to b typically look ?*

This naive question was of crucial importance at the time Feynman developed the so-called *path integral formulation* of quantum mechanics.

Already in this case, defining properly a "uniform measure" on paths was not an easy task. Yet, it had been mathematically settled prior to Feynman's work and corresponds to the well-known *Brownian motion.*

In some sense, the purpose of *quantum gravity* is to extend Feynman path integrals to Feynman integrals over *surfaces.* Physicists are particularly interested in such an extension, since this would provide a powerful tool to deal with the *quantization* of gravitation field theory, a notoriously hard problem. (1) With this background in mind, the problem of defining a proper mathematical object for a "uniformly chosen random metric on $\mathbb{S}^2$" thus corresponds to defining a two-dimensional analog of Brownian motion, i.e. a kind of *Brownian surface.*

(1) This approach towards the quantization of gravitation is called *loop quantum gravity.*

Even though physicists are primarily interested in the above continuum formulation of Question 1.1, a natural and very fruitful approach is to study an appropriate discretization of it and then to pass to the limit. This brings us to the next subsection.

1.2. Discretization of Question 1.1 and planar maps

In the one-dimensional setting, if one asks Question 1.2, it is not straightforward to come up right away with *Brownian motion*. But, if instead we start by discretizing Question 1.2, say by allowing random $1/n$-steps, then we end up with the model of *random walks*. Brownian motion is then obtained as the scaling limit (as $n \to \infty$) of these rescaled random walks.

It is thus tempting to apply the same strategy to Question 1.1, namely to find an appropriate discretization. Let us explain below a possible discretization which was used initially in the physics literature and was studied extensively recently among the mathematical community. See for example **[22]** and references therein. We will see in Subsection 1.3 that there are other ways to discretize Question 1.1 which lead to different universality classes [(2)], but the discretization below is in some sense the simplest and most natural one regarding the statement of Question 1.1.

The idea of the discretization we wish to introduce is to consider discrete graphs, with say n faces, which have the topology of a sphere $\mathbb{S}^2$ and for which the metric ρ_n will correspond (up to a rescaling factor) to the graph distance, i.e. $\rho_n := n^{-a} d_{gr}$ for some exponent $a > 0$. The exponent a will need to be well chosen as in the case of Random Walks, where space needs to be rescaled by $\sqrt{n}$ in order to obtain a limit. If we define our discretization in such a way that for each $n \geq 1$, there are finitely many such graphs, we can pick one uniformly at random (in the spirit of Question 1.1) which thus gives us a random metric space (M_n, ρ_n). We can then ask the question of the scaling limit of these random variables (M_n, ρ_n) as $n \to \infty$ in the space $(\mathbb{K}, d_{GH})$ of all isometry classes of compact metric spaces, endowed with the Gromov-Hausdorff distance d_{GH} [(3)]. The advantage of this setup is that $(\mathbb{K}, d_{GH})$ is a complete, separable, metric space (a Polish space) and is thus suitable to the analysis of convergences in law and so on. Note here, that even if one could prove that (M_n, ρ_n) converges to a limiting random object (M_∞, ρ_∞), it is not clear a priori that the topology is preserved at the scaling limit or in other words, it needs to be proved whether (M_∞, ρ_∞) a.s. has the topology of a sphere or not. If all these steps

(2) Similarly as Random Walks with non-L^2 steps converge to other Levy processes than Brownian Motion.

(3) Informally, if (E_1, d_1) and (E_2, d_2) are two compact metric spaces, then $d_{GH}(E_1, E_2)$ is computed as follows: we embed E_1 and E_2 isometrically into some larger metric space (E, d) and we compute using the common distance d the distance in the Hausdorff sense between the two embeddings. Then we take the infimum over the possible such embeddings. See **[22]**.

can be carried on, then this would give us a good candidate (M_∞, ρ_∞) for the random object used in Question 1.1.

Let us now introduce one specific discretization.

Definition 1.3 (planar map, following **[20]**). — A *planar map* M is a proper embedding of a finite and connected graph into the two-dimensional sphere $\mathbb{S}^2$, which is viewed up to orientation preserving homeomorphisms of $\mathbb{S}^2$ (i.e. up to "deformations"). Loops and multiple edges are allowed. The faces of M are identified with the connected components of $\mathbb{S}^2 \setminus M$ and the degree of a face $\mathbf{f}$ is defined as the number of edges incident to $\mathbf{f}$, with the additional rule that if both sides of an edge belong to the same face, this edge is counted twice.

Finally, for combinatorial reasons, it is often convenient to consider *rooted* planar maps, meaning that one particular oriented edge $\overrightarrow{e}$ is distinguished. The origin of that root edge $\overrightarrow{e}$ is called the *root vertex* $\varnothing$. See Figure 1 for an instance of a planar map where all faces happen to be squares.

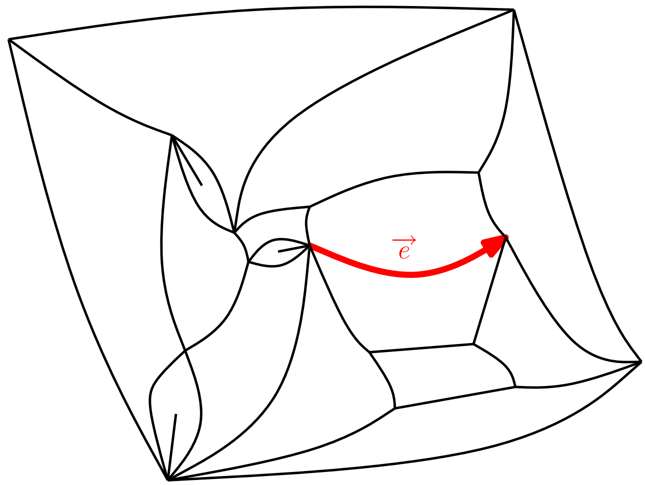

FIGURE 1. This is a planar map of the sphere $\mathbb{S}^2$ with exactly 17 squares (this includes the exterior square which is also in the sphere).

Definition 1.4 (p-angulations of the sphere). — For any integer $p \geq 3$, let $\mathbf{M}_n^p$ be the set of all rooted planar maps with n faces, where each face has degree p. The elements of $\mathbf{M}_n^p$ are called *rooted planar p-angulations*. ($p = 3$ corresponds to triangulations and $p = 4$ to quadrangulations).