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SMALL TIME EQUIVALENTS FOR THE DENSITY OF A PLANAR QUADRATIC LANGEVIN DIFFUSION

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ABSTRACT. — Exact small time equivalents for the density of the (heat kernel) semigroup, with a control of the error term, are obtained for a quadratic planar analogue of the Langevin diffusion, which is strictly hypoelliptic and non-Gaussian, and hence of a different nature from the known Riemannian, sub-Riemannian and linear-Gaussian cases. Two regimes are considered, an unscaled and a scaled one, where both can be seen as natural extensions beyond the degenerate Langevin-Gaussian framework. The result for the scaled regime seems to be the first such one in a non-Gaussian strictly hypoelliptic framework. The method is half-probabilistic, half-analytic.

RÉSUMÉ (*Equivalents en temps petit pour la densité d'une diffusion de Langevin quadratique plane*). — Cet article fournit des équivalents exacts en temps petit, avec contrôle du terme d'erreur, relatifs à la densité (noyau de la chaleur) du semi-groupe associé à une diffusion quadratique plane, analogue non gaussien de la diffusion de Langevin. Dans ce cadre strictement hypoelliptique non gaussien, différent des cadres sous-riemannien et gaussien (linéaire), le régime de base et un régime rééchelonné sont considérés, qui sont tous deux des prolongements naturels du cas dégénéré Langevin-gaussien. L'étude du régime rééchelonné semble la première de ce type dans un tel cadre. La méthode suivie est mi-probabiliste mi-analytique.

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1. Introduction

The problem of estimating the heat kernel, or the density of a diffusion, particularly as time goes to zero, has been extensively studied for a long time, firstly in the elliptic case, and then largely solved and understood in the sub-Riemannian case too. We only reference articles [20, 2, 4, 5, 17], and the existence of other works on that subject by Azencott, Molchanov and Bismut, quoted in [4].

To summary roughly, a very classical question addresses the asymptotic behavior (as $s \searrow 0$) of the density $p_s(x,y)$ of the diffusion (x_s) solving a Stratonovich stochastic differential equation

$$x_s = x + \sum_{j=1}^{k} \int_0^s V_j(x_\tau) \circ dW_\tau^j + \int_0^s V_0(x_\tau)\, d\tau\,,$$

where the smooth vector fields V_j are supposed to satisfy a Hörmander condition.

The elliptic case (when $V_1,\ldots,V_k$ span the whole tangent space everywhere) being very well understood for a long time [20, 2], the studies focused then on the sub-elliptic case, that is to say, when the strong Hörmander condition (that the Lie algebra generated by the fields $V_1,\ldots,V_k$ has maximal rank everywhere) is fulfilled. In that case these fields generate a sub-Riemannian distance $d(x,y)$, defined as in control theory, by considering only C^1 paths whose tangent vectors are spanned by them. Then the wanted asymptotic expansion tends to have the following Gaussian-like form:

$$p_\varepsilon(x,y) = \varepsilon^{-d/2} \exp\bigl(-d(x,y)^2/(2\varepsilon)\bigr)\Biggl(\sum_{\ell=0}^{n} \gamma_\ell(x,y)\,\varepsilon^\ell + \mathcal{O}(\varepsilon^{n+1})\Biggr) \tag{1}$$

for any $n \in \mathbb{N}^*$, with smooth γ_ℓ's and $\gamma_0 > 0$, provided x,y are not conjugate points (and uniformly within any compact set which does not intersect the cut-locus). See in particular ([4, theorem 3.1]). Note that the condition of remaining outside the cut-locus is necessary here, as shown in particular by [5].

The methods used to get this or a similar result have been of a different nature. In [4], G. Ben Arous proceeds by expanding the flow associated to the diffusion (in this direction, see also [7]) and using a Laplace method applied to the Fourier transform of x_s, then inverted by means of Malliavin's calculus (with a deterministic Malliavin matrix).

The strictly hypoelliptic case, i.e., when only the weak Hörmander condition (requiring the use of the drift vector field V_0 to recover the full tangent space) is fulfilled, remains much more problematic, and thus is rarely addressed. There is a priori no reason that the asymptotic behavior of $p_s(x,y)$ remains of the Gaussian-like type (1). Indeed this already fails for the mere Gaussian Langevin process $\bigl(\omega_s, \int_0^s \omega_\tau\, d\tau\bigr)$: the missing sub-Riemannian distance must be replaced by a time-dependent (actually Carnot-Carathéodory) distance $d_\varepsilon\big((\dot x, x);(\dot y, y)\big)$ which presents some degeneracy in one direction, namely the missing $d\big((\dot x, x);(\dot y, y)\big)^2/(2\varepsilon)$ must be replaced by

$$\begin{aligned}&\frac{6}{\varepsilon^3}\left|(x-y)-\frac{\varepsilon}{2}\,(\dot x-\dot y)\right|^2+\frac{1}{2\varepsilon}\left|\dot x-\dot y\right|^2\\ &=\frac{1}{2\varepsilon}\Big(\left|\dot x-\dot y\right|^2+\frac{12}{\varepsilon^2}\left|(x-y)-\frac{\varepsilon}{2}\,(\dot x-\dot y)\right|^2\Big),\end{aligned}$$

and actually, for any $\varepsilon>0$ and $\dot x, x, \dot y, y\in\mathbb{R}^d$ we have

$$\begin{aligned}(2)\qquad &p_\varepsilon\big((\dot x, x);(\dot y, y)\big)\\ &=\frac{3^{d/2}}{\pi^d\,\varepsilon^{2d}}\,\exp\left[-\frac{|\dot x-\dot y|^2+12\left|(x-y)-\varepsilon\,(\dot x-\dot y)/2\right|^2/\varepsilon^2}{2\,\varepsilon}\right].\end{aligned}$$

As in this Gaussian-Euclidean Langevin case this expression is actually an exact one, and holds not only asymptotically, we then have

$$\begin{aligned}(3)\qquad &p_\varepsilon\big((\dot x, \varepsilon x);(\dot y, \varepsilon y)\big)\\ &=\frac{3^{d/2}}{\pi^d\,\varepsilon^{2d}}\,\exp\left[-\frac{|\dot x-\dot y|^2+12\left|(x-y)-\frac{1}{2}\,(\dot x-\dot y)\right|^2}{2\,\varepsilon}\right],\end{aligned}$$

for any $\varepsilon>0$ and $\dot x, x, \dot y, y\in\mathbb{R}^d$. Thus, in this scaled formulation, in the energy we recover a true, time-independent squared distance. So that, referring to the Riemannian and sub-Riemannian cases, there is no clear reason a priori to favour the unscaled version (2) to the scaled version (3). We shall emphasize this point of view below.

Barilari and Paoli [3] considers a general Gaussian hypoelliptic n-dimensional diffusion (X_t), solving a linear equation $dX_t = A\,X_t\,dt + B\,dW_t$, for a d-dimensional Brownian motion W. Taking advantage of the explicit exact expression for the heat kernel p_t which is computable in such a linear Gaussian case, the authors provide the full small time asymptotics for p_t.

Paoli [18] analyzes the small time asymptotics on the diagonal, relative to the heat kernel of a manifold-valued strictly hypoelliptic diffusion, in the spirit of previous works by Ben Arous and Léandre.

See also [9] and [19] for non-curved, strictly hypoelliptic, perturbed cases where Langevin-like estimates hold (without precise asymptotics), roughly having the following Li-Yau-like form:

$$\begin{aligned}(4)\qquad C^{-1}\,\varepsilon^{-N}\,e^{-C\,d_\varepsilon(x_\varepsilon,y)^2}&\le p_\varepsilon(x,y)\\ &\le C\,\varepsilon^{-N}\,e^{-C^{-1}\,d_\varepsilon(x_\varepsilon,y)^2},\quad \text{for } 0<\varepsilon<\varepsilon_0\,.\end{aligned}$$

In [11] a small time asymptotics of a simple model was only partly computed, similar to the one analyzed below but in five dimensions, in the specific off-diagonal regime of a dominant normalized Gaussian contribution. Thus the energy term appeared as given by the same squared time-dependent distance as in the Langevin case, the strictly second chaos coordinate appearing only in the off-exponent term, as a perturbative contribution.

A stronger interest lies in a significant strictly hypoelliptic diffusion, namely the relativistic diffusion, first constructed over Minkowski's space (see [10, 13]). It makes sense over a generic smooth Lorentzian manifold as well, see [12]. In the simplest case of Minkowski's space $\mathbb{R}^{1,d}$, it consists of the pair $(\dot{\xi}_s, \xi_s) \in \mathbb{H}^d \times \mathbb{R}^{1,d} \equiv T^1_+\mathbb{R}^{1,d}$ (parametrized by its proper time s, and analogous to a Langevin process), where the velocity $(\dot{\xi}_s)$ is a hyperbolic Brownian motion. Note that even there, a curvature constraint must be taken into account, namely that of the mass shell $\mathbb{H}^d$, at the heart of this framework.

This (Dudley) relativistic diffusion, even restricted to 3 dimensions which already contain the essence of the difficulty, constitutes a significant example, altogether explicit, physical and not too complicated, allowing a priori to progress towards the understanding of a more generic, but less accessible to begin with, strictly hypoelliptic (degenerate) case. However even this apparently simple example proves to be very delicate to analyze, regarding the small time asymptotics.

The present work handles a simpler example, also of the Langevin type, hence strictly hypoelliptic, degenerate but non-Gaussian. It also pertains to the Euclidean case of what [1] recently considered and called "kinetic Brownian motion", on a Riemannian manifold. In this setting we obtain the exact small time ($\varepsilon \searrow 0$) equivalent for the heat kernel density $p_\varepsilon(0; x)$, together with a control of the error term, for constant x and also in an interesting scaled case where x exhibits some dependence on ε. To the author's knowledge, the latter is the first example of a result of this type in a non-Gaussian strictly hypoelliptic framework. It reveals a different nature from the known Gaussian framework, see Remark 2.2 below.

The present work was influenced by a beautiful article [4], which decisively handled the off-cut-locus (hence in particular off-diagonal) generic sub-Riemannian framework, as far as the small-time asymptotics of the heat kernel is considered. Thus the strategy adopted below starts as the strategy followed by G. Ben Arous. However the present purpose is to deal with a strictly hypoelliptic situation, to which [4] does not apply. A main obstacle to handle a strictly hypoelliptic framework is the lack of sub-Riemannian distance. For that reason, a strategy adapted to such a degenerate framework can only partially follow the method of [4].

As the author is not yet ready to handle a generic strictly hypoelliptic framework, and actually doubts that a generic unified treatment (or even, unified generic result) be possible (different strictly hypoelliptic frameworks could produce different types of results; the present one already differs notably from the classical Gaussian Langevin case regarding the scaled energy, see Remark 2.2

below), the focus here is on a simple first example, which allows explicit computing of the Fourier transform of the heat kernel, and then concentrating on some finite-dimensional oscillatory integral. On the contrary, the choice of a 2-dimensional framework is unessential, but avoids even heavier notation and computations which higher dimensions would call for. According to the above remark about the Gaussian-Euclidean Langevin case, we consider both the unscaled and the scaled asymptotics, and the latter appears here as the most interesting, maybe indicating that the known settings are more interestingly extended to the strictly hypoelliptic framework in this way.

Besides, focusing on the present relatively simple example allows handling of the "pseudo-cut-locus" case, the analogue of conjugate points; which is delicate, even in the sub-Riemannian framework: the cut-locus constitutes a real difficulty in that case, see for example [5], and its case does not seem to be generically solved in a sub-elliptic framework. Furthermore the choice of a relatively simple particular framework allows expressing of all coefficients of the wanted equivalents, together with a control of the error, which will likely be out of reach in an even slightly more generic framework; however even in the present setting most functions that come into the scaled result remain implicit. In order to keep the already heavy enough computations within reasonable bounds, we focus on the asymptotics for the process started from 0.

Whereas the unscaled regime was already considered by V. Kolokoltsov in his book [16], by different, purely analytical means, see Remark 2.8 below, the scaled study seems to be the first one of this type in such a framework.

Organization of the content. In Section 2 the strictly hypoelliptic diffusion under consideration is described, and the central results, relating to both the unscaled case and the scaled case, are gathered in Theorem 2.1. Corollary 2.7 states that the squared Carnot-Carathéodory pseudo-distance yields the right exponent in the unscaled asymptotics, as in the (sub-)Riemannian [4] and Gaussian [3] frameworks.

Section 3 develops the leading strategy of the proof, which begins as that of [4]: the first main tool is a Fourier-Parseval expression for the density of the heat kernel under consideration, see Proposition 3.4. Here the lack of metric and geodesics forbids the use of a geodesic tube as in [4]. This is replaced by another key tool, which is the explicit computation of the Fourier transform, which is possible due to the choice of a Langevin-like diffusion of a quadratic type. The latter explains the reason for this choice, and why the present strategy could hardly be extended to non-quadratic examples, see Remark 2.5 below. The expression for the heat kernel density obtained in this way contains an oscillatory integral which is not computable, but which is no longer infinite-dimensional.