Panoramas & Synthèses
25, 2008, p. 1–119

AN INVITATION TO RANDOM SCHRÖDINGER OPERATORS

by

Werner Kirsch
with an Appendix by Frédéric Klopp

Abstract. – These lecture notes try to give some of the basics of random Schrödinger operators. They are meant for nonspecialists and require only minor previous knowledge about functional analysis and probability theory. Nevertheless this survey includes complete proofs of Lifshitz tails and Anderson localization. An appendix written by F. Klopp is devoted to the Aizenman-Molchanov proof of Anderson localization.

***Résumé* (Une invitation aux opérateurs de Schrödinger aléatoires).** – Ces notes essayent de présenter les bases de la théorie des opérateurs de Schrödinger aléatoires. Elles sont destinées à un public très large et ne requièrent que des connaissances minimales en analyse fonctionnelle et en théorie des probabilités. Néanmoins, on y donne une démonstration complète des asymptotiques de Lifshitz et de la localisation d'Anderson. Ces notes ont été augmentées d'un appendice écrit par F. Klopp, appendice présentant une preuve de la localisation d'Anderson suivant les idées d'Aizenman et Molchanov.

1. Preface

In these lecture notes I try to give an introduction to (some part of) the basic theory of random Schrödinger operators. I intend to present the field in a rather self contained and elementary way. It is my hope that the text will serve as an introduction to random Schrödinger operators for students, graduate students and researchers who have not studied this topic before. If some scholars who are already acquainted with random Schrödinger operators might find the text useful as well I will be even more satisfied.

Only a basic knowledge in Hilbert space theory and some basics from probability

---

2000 Mathematics Subject Classification. – 82B44, 35J10, 47B80, 60H25, 81Q10.
Key words and phrases. – Random Schrödinger operators, Anderson localization, density of states, Lifshitz asymptotics.

theory are required to understand the text (see the Notes below). I have restricted the considerations in this text almost exclusively to the Anderson model, i.e. to random operators on the Hilbert space $\ell^2(\mathbb{Z}^d)$. By doing so I tried to avoid many of the technical difficulties that are necessary to deal with in the continuous case (i.e. on $L^2(\mathbb{R}^d)$). Through such technical problems sometimes the main ideas become obscured and less transparent.

The theory I present is still not exactly easy staff. Following Einstein's advice, I tried to make things as easy as possible, but not easier.

The author has to thank many persons. The number of colleagues and friends I have learned from about mathematical physics and especially disordered systems is so large that it is impossible to mention a few without doing injustice to many others. A lot of the names can be found as authors in the list of references. Without these persons the writing of this review would have been impossible.

A colleague and friend I *have* to mention though is Frédéric Klopp who organized a summer school on Random Schrödinger operators in Paris in 2002. My lectures there were the starting point for this review. I have to thank Frédéric especially for his enormous patience when I did not obey the third, forth, ..., deadline for delivering the manuscript.

It is a great pleasure to thank Bernd Metzger for his advice, for many helpful discussions, for proofreading the manuscript, for helping me with the text and especially with the references and for many other things.

Last, not least I would like to thank Jessica Langner, Riccardo Catalano and Hendrik Meier for the skillful typing of the manuscript, for proofreading and for their patience with the author.

Notes and Remarks. – For the spectral theory needed in this work we recommend [**107**] or [**131**]. We will also need the min-max theorem (see [**105**]).

The probabilistic background we need can be found e.g. in [**88**] and [**87**].

For further reading on random Schrödinger operators we recommend [**69**] for the state of the art in multiscale analysis. We also recommend the textbook [**121**]. A modern survey on the density of states is [**62**].

2. Introduction: Why random Schrödinger operators?

2.1. The setting of quantum mechanics. – A quantum mechanical particle moving in d-dimensional space is described by a vector ψ in the Hilbert space $L^2(\mathbb{R}^d)$. The time evolution of the state ψ is determined by the *Schrödinger operator*

$$H = H_0 + V \tag{2.1}$$

acting on $L^2(\mathbb{R}^d)$. The operator H_0 is called the *free operator*. It represents the kinetic energy of the particle. In the absence of magnetic fields it is given by the Laplacian

$$H_0 = -\frac{\hbar^2}{2m}\,\Delta = -\frac{\hbar^2}{2m}\sum_{\nu=1}^{d}\frac{\partial^2}{\partial {x_\nu}^2}. \tag{2.2}$$

The physics of the system is encoded in the potential V which is the multiplication operator with the function $V(x)$ in the Hilbert space $L^2(\mathbb{R}^d)$. The function $V(x)$ is the (classical) potential energy. Consequently, the forces are given by

$$F(x) = -\nabla V(x).$$

In the following we choose physical units in such a way that $\frac{\hbar^2}{2m} = 1$ since we are not interested in the explicit dependence of quantities on $\hbar$ or m. The time evolution of the state ψ is obtained from the time dependent Schrödinger equation

$$i\,\frac{\partial}{\partial t}\,\psi = H\,\psi. \tag{2.3}$$

By the spectral theorem for self adjoint operators equation (2.3) can be solved by

$$\psi(t) = e^{-itH}\psi_0 \tag{2.4}$$

where ψ_0 is the state of the system at time $t = 0$.

To extract valuable information from (2.4) we have to know as much as possible about the spectral theory of the operator H and this is what we try to do in this text.

2.2. Random Potentials. – In this review we are interested in random Schrödinger operators. These operators model disordered solids. Solids occur in nature in various forms. Sometimes they are (almost) totally ordered. In crystals the atoms or nuclei are distributed on a periodic lattice (say the lattice $\mathbb{Z}^d$ for simplicity) in a completely regular way. Let us assume that a particle (electron) at the point $x \in \mathbb{R}^d$ feels a potential of the form $q\,f(x-i)$ due to an atom (or ion or nucleus) located at the point $i \in \mathbb{Z}^d$. Here, the constant q, the charge or coupling constant in physical terms, could be absorbed into the function f. However, since we are going to vary this quantity from atom to atom later on, it is useful to write the potential in the above way. Then, in a regular crystal our particle is exposed to a total potential

$$V(x) = \sum_{i\in\mathbb{Z}^d} q\,f(x-i). \tag{2.5}$$

We call the function f the *single site potential* to distinguish it from the total potential V. The potential V in (2.5) is periodic with respect to the lattice $\mathbb{Z}^d$, i. e. $V(x-i) = V(x)$ for all $x \in \mathbb{R}^d$ and $i \in \mathbb{Z}^d$. The mathematical theory of Schrödinger operators with periodic potentials is well developed (see e.g. [**34**], [**105**]). It is based on a thorough analysis of the symmetry properties of periodic operators. For example, it is known that such operators have a spectrum with band structure, i.e. $\sigma(H) =$

$\bigcup_{n=0}^{\infty}[a_n, b_n]$ with $a_n < b_n \leq a_{n+1}$. This spectrum is also known to be absolutely continuous.

Most solids do not constitute ideal crystals. The positions of the atoms may deviate from the ideal lattice positions in a non regular way due to imperfections in the crystallization process. Or the positions of the atoms may be completely disordered as is the case in amorphous or glassy materials. The solid may also be a mixture of various materials which is the case for example for alloys or doped semiconductors. In all these cases it seems reasonable to look upon the potential as a random quantity.

For example, if the material is a pure one, but the positions of the atoms deviate from the ideal lattice positions randomly, we may consider a random potential of the form

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q\, f\left(x - i - \xi_i(\omega)\right). \tag{2.6}$$

Here the ξ_i are random variables which describe the deviation of the "i^{th}" atom from the lattice position i. One may, for example assume that the random variables ξ_i are independent and identically distributed. We have added a subscript ω to the potential V to make clear that V_ω depends on (unknown) random parameters.

To model an amorphous material like glass or rubber we assume that the atoms of the material are located at completely random points η_i in space. Such a random potential may formally be written as

$$V_\omega(x) = \sum_{i \in \mathbb{Z}^d} q\, f(x - \eta_i). \tag{2.7}$$

To write the potential (2.7) as a sum over the lattice $\mathbb{Z}^d$ is somewhat misleading, since there is, in general, no natural association of the η_i with a lattice point i. It is more appropriate to think of a collection of random points in $\mathbb{R}^d$ as a random point measure. This representation emphasizes that any ordering of the η_i is completely artificial.

A *counting measure* is a Borel measure on $\mathbb{R}^d$ of the form $\nu = \sum_{x \in M} \delta_x$ with a countable set M without (finite) accumulation points. By a *random point measure* we mean a mapping $\omega \mapsto \mu_\omega$, such that μ_ω is a counting measure with the property that the function $\omega \mapsto \mu_\omega(A)$ is measurable for any bounded Borel set A. If $\nu = \nu_\omega$ is the random point measure $\nu = \sum_i \delta_{\eta_i}$ then (2.7) can be written as

$$V_\omega(x) = \int_{\mathbb{R}^d} q\, f(x - \eta)\, d\nu(\eta). \tag{2.8}$$

The most frequently used example of a random point measure and the most important one is the *Poisson random measure* μ_ω. Let us set $n_A = \mu_\omega(A)$, the number of random points in the set A. The Poisson random measure can be characterized by the following specifications

- The random variables n_A and n_B are independent for disjoint (measurable) sets A and B.
- The probability that $n_A = k$ is equal to $\frac{|A|^k}{k!}e^{-|A|}$, where $|A|$ is the Lebesgue measure of A.

A random potential of the form (2.8) with the Poisson random measure is called the *Poisson model.*

The most popular model of a disordered solid and the best understood one as well is the *alloy-type potential* (see (2.9) below). It models an unordered alloy, i.e. a mixture of several materials the atoms of which are located at lattice positions. The type of atom at the lattice point i is assumed to be random. In the model we consider here the different materials are described by different charges (or coupling constants) q_i. The total potential V is then given by

$$V_\omega(x) = \sum_{i\in\mathbb{Z}^d} q_i(\omega)\, f(x-i). \tag{2.9}$$

The q_i are random variables which we assume to be independent and identically distributed. Their range describes the possible values the coupling constant can assume in the considered alloy. The physical model suggests that there are only finitely many values the random variables can assume. However, in the proofs of some results we have to assume that the distribution of the random variables q_i is continuous (even absolutely continuous) due to limitations of the mathematical techniques. One might argue that such an assumption is acceptable as a purely technical one. On the other hand one could say we have not understood the problem as long as we can not handle the physically relevant cases.

For a given ω the potential $V_\omega(x)$ is a pretty complicated "normal" function. So, one may ask: What is the advantage of "making it random"?

With the introduction of random variables we implicitly change our point of view. From now on we are hardly interested in properties of H_ω for a single given ω. Rather, we look at "typical" properties of H_ω. In mathematical terms, we are interested in results of the form: The set of all ω such that H_ω has the property $\mathcal{P}$ has probability one. In short: $\mathcal{P}$ holds for $\mathbb{P}$-almost all ω (or $\mathbb{P}$-almost surely). Here $\mathbb{P}$ is the probability measure on the underlying probability space.

In this course we will encounter a number of such properties. For example we will see that (under weak assumptions on V_ω), there is a closed, nonrandom (!) subset Σ of the real line such that $\Sigma = \sigma(H_\omega)$, the spectrum of the operator H_ω, $\mathbb{P}$-almost surely.

This and many other results can be proven for various types of random Schrödinger operators. In this lecture we will restrict ourselves to a relatively simple system known as the Anderson model. Here the Hilbert space is the sequence space $\ell^2(\mathbb{Z}^d)$ instead of $L^2(\mathbb{R}^d)$ and the free operator H_0 is a finite-difference operator rather than the Laplacian. We will call this setting the *discrete case* in contrast to Schrödinger operators on $L^2(\mathbb{R}^d)$ which we refer to as the *continuous case.* In the references the reader may find papers which extend results we prove here to the continuous setting.