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MULTISCALE ANALYSIS AND LOCALIZATION OF RANDOM OPERATORS

by

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Abstract. – This paper is devoted to the method of multiscale analysis in the study of localization of random operators. The method is developed not only for Schrödinger operators, but also for acoustic, Maxwell or elastic operators. It is one of the basic techniques to obtain results on the localized regimes for continuous random operators.

***Résumé* (Analyse multi-échelle et localisation pour des opérateurs aléatoires).** – Cet article est consacré à l'analyse multi-échelle et à la localisation pour des opérateurs aléatoires. Cette analyse y est développée non seulement pour les opérateurs de Schrödinger mais aussi pour des modèles acoustiques ou de Maxwell. C'est l'un des principaux outils menant à des résultats sur la localisation pour ces opérateurs.

1. Introduction

In his seminal 1958 article [**5**], Anderson argued that for a simple Schrödinger operator in a disordered medium, "at sufficiently low densities transport does not take place; the exact wave functions are localized in a small region of space." This phenomenon, known as Anderson localization, originally studied in the context of quantum mechanical electrons in random media (e.g., [**84**]), was later found relevant also in the context of classical waves in random media (e.g., [**6, 75, 51, 52**]), where it was observed in light waves in an experiment conducted by Wiersma et al [**88**].

Anderson localization was initially given a spectral interpretation: pure point spectrum with exponentially decaying eigenstates (exponential localization). But the intuitive physical notion of localization has also a dynamical interpretation: the moments of a wave packet, initially localized both in space and in energy, should remain uniformly bounded under time evolution. (Dynamical localization implies pure point

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spectrum, but the converse is not true.) Although exponential localization has sometimes been called Anderson localization, we will use Anderson localization in a broader sense, since it can be argued the circle of ideas regarding localization, originating from [**5**], include the physical notion of dynamical localization.

Localization for random operators was first established in the celebrated paper by Gol'dsheid, Molchanov and Pastur [**48**] for a one dimensional continuous random Schrödinger operator. Their method was extended to other one and quasi-one (the strip) dimensional random Schrödinger operators [**71, 12, 72**]. But the multi-dimensional case required new methods.

The method with the wider applicability has been the multiscale analysis, a technique initially developed by Fröhlich and Spencer [**34**] and Fröhlich, Martinelli, Spencer and Scoppolla [**33**], and simplified by von Dreifus [**25**] and von Dreifus and Klein [**26**]. (For the multiscale analysis per se, see also [**73, 82, 27, 56, 49, 66, 15, 30, 55, 54, 70, 83, 41, 43**], for applications see also [**13, 64, 63, 65, 67, 29, 28, 16, 31, 86, 8, 7, 79, 19, 59, 23, 32, 69, 89, 21, 85, 61, 46, 42, 44**].) Although it originally only gave exponential localization [**33, 24, 81, 26, 15**], it was later shown to also yield dynamical localization by Germinet and De Bièvre [**36**], strong dynamical localization for moments up to some finite order by Damanik and Stollman [**22**], and strong dynamical localization (up to all orders) in the Hilbert-Schmidt norm by Germinet and Klein [**41**]. The latest version of the multiscale analysis, the bootstrap multiscale analysis of Germinet and Klein [**41**], built out of four different multiscale analyses, yields exponential localization, semi-uniformly localized eigenfunctions (SULE), and sub-exponential decay of the expectation of the kernel of the evolution operator.

The other successful method for proving localization in the multi-dimensional case is the fractional moment method introduced by Aizenman and Molchanov [**3, 1, 4**], which has just been extended to the continuum by Aizenman et al [**2**]. It yields exponential decay for the expectation of the kernel of the evolution operator, but it requires that the conditional expectation of certain random variables have bounded densities.

In these lectures we discuss the method of multiscale analysis in the study of localization of random operators. A random medium will be modeled by a ergodic random self-adjoint operator. In Section 2 we discuss the most important random operators: random Schrödinger operators, random Landau Hamiltonians, and random classical wave operators (Maxwell, acoustic, elastic). In Section 3 we discuss several definitions of localization from both the spectral and dynamical points of view. In Section 4 we describe the properties of random operators required by the multiscale analysis. In Section 5 we state and discuss the bootstrap multiscale analysis plus the four multiscale analyses used in its proof. In Section 6 we prove exponential and dynamical localization from the multiscale analysis. In Section 7 we show how to perform a multiscale analysis; we give a complete proof of the Dreifus-Klein multiscale analysis in the continuum.

These lectures were written in 2002. Since then Bourgain and Kenig [**11**] proved localization in the continuous Anderson-Bernoulli model, using a multiscale analysis. The Wegner estimate is established in the multiscale analysis using "free sites" and a new quantitative version of unique continuation which gives a lower bound on eigenfunctions. Since their Wegner estimate has weak probability estimates and the underlying random variables are discrete, they also introduced a new method to prove Anderson localization from estimates on the finite-volume resolvents given by a single-energy multiscale analysis. The new method does not use spectral averaging as in [**15, 24, 81**], which requires random variables with bounded densities. It is also not an energy-interval multiscale analysis as in [**26, 33**], which requires better probability estimates. Subsequently, Germinet, Hislop and Klein [**37, 39, 38**] proved localization for Schrödinger operators with Poisson random potential, using a multiscale analysis that exploits the probabilistic properties of Poisson point processes to control the randomness of the configurations, and at the same time allows the use of the new ideas introduced by Bourgain and Kenig.

2. Random operators

Quantum and classical waves in random media are modeled by random self-adjoint operators on either $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x; \mathbb{C}^n)$ or $\ell^2(\mathbb{Z}^d; \mathbb{C}^n)$. Examples include:

– Random Schrödinger operators:
  - The Anderson model:

$$H_\omega = -\Delta + V_\omega \quad \text{on} \quad \ell^2(\mathbb{Z}^d), \tag{2.1}$$

where Δ is the finite difference Laplacian and $\{V_\omega(x);\ x \in \mathbb{Z}^d\}$ are independent identically distributed bounded random variables. (E.g., [**71, 34, 72, 33, 13, 74, 64, 20, 26, 82, 63, 56, 49, 3, 1, 29, 58, 57, 79, 4, 87, 68**].)

  - Anderson Hamiltonians on the continuum:

$$H_\omega = -\Delta + V_{\mathrm{per}} + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{2.2}$$

where Δ is the Laplacian operator, V_{per} is a periodic potential (by rescaling we take the period to be one) of the form $V_{\mathrm{per}} = V_{\mathrm{per}}^{(1)} + V_{\mathrm{per}}^{(2)}$, with $V_{\mathrm{per}}^{(i)}$, $i = 1, 2$, periodic with period one, $0 \le V_{\mathrm{per}}^{(1)} \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d, \mathrm{d}x)$, $V_{\mathrm{per}}^{(2)}$ relatively form-bounded with respect to $-\Delta$ with relative bound < 1, and V_ω a random potential of the form

$$V_\omega(x) = \sum_{i \in \frac{1}{q}\mathbb{Z}^d} \omega_i\, u(x - i), \tag{2.3}$$

where $q \in \mathbb{N}$, $\omega = \{\omega_i;\ i \in \frac{1}{q}\mathbb{Z}^d\}$ are independent identically distributed bounded random variables, u is a real valued measurable function with compact support, $u \in \mathrm{L}^p(\mathbb{R}^d, \mathrm{d}x)$ with $p > \frac{d}{2}$ if $d \geq 2$ and $p = 2$ if $d = 1$. (E.g., [**73, 66, 65, 15, 67, 8, 55, 54, 36, 83, 41, 22, 21, 69, 89, 46, 43, 42, 44, 2**].)

– Random Landau Hamiltonians:

$$H_\omega = H_0 + V_\omega \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x), \tag{2.4}$$

where $H_0 = (-i\nabla - A)^2$, $A = \frac{B}{2}(x_2, -x_1)$ with $B > 0$, and the random potential V_ω is as in (2.3) with $q = 1$ and $u(x)$ bounded. (See [**16, 86, 7, 43**].)

– Random classical wave operators:

- Maxwell operators in random media:

$$H_\omega = \frac{1}{\sqrt{\mu_\omega(x)}} \nabla\times \frac{1}{\varepsilon_\omega(x)} \nabla\times \frac{1}{\sqrt{\mu_\omega(x)}} \quad \text{on} \quad \mathrm{L}^2\left(\mathbb{R}^3, \mathrm{d}x; \mathbb{C}^3\right) \tag{2.5}$$

where $\nabla\times$ is the operator given by the curl, $\varepsilon_\omega(x)$ is the random dielectric constant and $\mu_\omega(x)$ is the random magnetic permeability. We take

$$\varepsilon_\omega(x) \quad = \quad \varepsilon_0(x)\gamma_\omega(x) \ \text{, with } \ \gamma_\omega(x) = 1 + \sum_{i \in \frac{1}{q}\mathbb{Z}^3} \omega_i u(x - i), \tag{2.6}$$

$$\mu_\omega(x) \quad = \quad \mu_0(x)\beta_\omega(x) \ \text{, with } \ \beta_\omega(x) = 1 + \sum_{i \in \frac{1}{q}\mathbb{Z}^3} \omega_i v(x - i), \tag{2.7}$$

where $q \in \mathbb{N}$, $\omega = \{\omega_i;\ i \in \frac{1}{q}\mathbb{Z}^d\}$ are independent identically distributed bounded random variables taking values in the interval $[-1, 1]$, $\varepsilon_0(x)$ and $\mu_0(x)$ are periodic measurable functions (by rescaling we take the period to be one), such that $0 < \varepsilon_- \leq \varepsilon(x) \leq \varepsilon_+ < \infty$ and $0 < \mu_- \leq \mu(x) \leq \mu_+ < \infty$ for some constants $\varepsilon_\pm$ and $\mu_\pm$, $u(x)$ and $v(x)$ are nonnegative measurable real valued functions with compact support, such that

$$0 \leq U_- \leq U(x) \equiv \sum_{i \in \frac{1}{q}\mathbb{Z}^3} u_i(x) \quad \leq \quad U_+ < \infty, \tag{2.8}$$

$$0 \leq V_- \leq V(x) \equiv \sum_{i \in \frac{1}{q}\mathbb{Z}^3} v_i(x) \quad \leq \quad V_+ < \infty, \tag{2.9}$$

for some constants $U_\pm$ and $V_\pm$, with $U_- + V_- > 0$ and $\max\{U_+, V_+\} < 1$. (See [**28, 31, 59, 19, 60, 61**].)

- Acoustic operators in random media:

$$H_\omega = \frac{1}{\sqrt{\kappa_\omega(x)}} \nabla^* \frac{1}{\rho_\omega(x)} \nabla \frac{1}{\sqrt{\kappa_\omega(x)}} \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x), \tag{2.10}$$

where ∇ is the gradient operator, and the random compressibility $\kappa_\omega(x)$ and the random mass density $\varrho_\omega(x)$ are of the same form as $\varepsilon_\omega(x)$ and $\mu_\omega(x)$ in (2.6) and (2.7). (See [**28, 30, 19, 60, 61**]).

- Elastic operators in random media:

$$H_\omega = \frac{1}{\sqrt{\rho_\omega(x)}} \left\{\nabla\left(\lambda_\omega(x) + 2\mu_\omega(x)\right)\nabla^* + \nabla \times \mu_\omega(x)\nabla\times\right\} \frac{1}{\sqrt{\rho_\omega(x)}} \tag{2.11}$$

on $\mathrm{L}^2\left(\mathbb{R}^3, \mathrm{d}x; \mathbb{C}^3\right)$, where the mass density $\rho_\omega(x)$, and the Lamé moduli $\lambda_\omega(x)$ and $\mu_\omega(x)$ are of the same form as $\varepsilon_\omega(x)$ and $\mu_\omega(x)$ in (2.6) and (2.7). (See [**60, 61**]).

In all these examples the random operator H_ω is a $\mathbb{Z}^d$-ergodic random self-adjoint operator H_ω on a Hilbert space $\mathcal{H}$, where ω belongs to a set Ω with a probability measure $\mathbb{P}$ and expectation $\mathbb{E}$, and either $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x; \mathbb{C}^n)$ ("on the continuum") or $\mathcal{H} = \ell^2(\mathbb{Z}^d; \mathbb{C}^n)$ ("on the lattice"). They all satisfy the following definition.

Definition 2.1. – *An ergodic random operator is a $\mathbb{Z}^d$-ergodic measurable map H_ω from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ (with expectation $\mathbb{E}$) to self-adjoint operators on either $\mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x; \mathbb{C}^n)$ or $\ell^2(\mathbb{Z}^d; \mathbb{C}^n)$.*

By measurability of H_ω we mean that the mappings $\omega \to f(H_\omega)$ are weakly (and hence strongly) measurable for all bounded Borel measurable functions f on $\mathbb{R}$. (See [**53**], [**14**, Section V.1] for more details.) Random operators may be defined without any ergodicity requirement, ergodicity being an extra requirement, but since we will be dealing only with $\mathbb{Z}^d$-ergodic random operators, we included it in the definition for convenience. We recall that H_ω is $\mathbb{Z}^d$-ergodic if there exists a group representation of $\mathbb{Z}^d$ by an ergodic family $\{\tau_y;\ y \in \mathbb{Z}^d\}$ of measure preserving transformations on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$U(y)H_\omega U(y)^* = H_{\tau_y(\omega)} \text{ for all } y \in \mathbb{Z}^d, \tag{2.12}$$

where $U(y)$ is the unitary operator given by translation: $(U(y)f)(x) = f(x-y)$. (Note that for Landau Hamiltonians translations are replaced by magnetic translations.)

An important consequence of ergodicity is that there exists a nonrandom set Σ such that $\sigma(H_\omega) = \Sigma$ with probability one, where $\sigma(A)$ denotes the spectrum of the operator A. In addition, the decomposition of $\sigma(H_\omega)$ into pure point spectrum $\sigma_{pp}(H_\omega)$, absolutely continuous spectrum $\sigma_{ac}(H_\omega)$, and singular continuous spectrum $\sigma_{sc}(H_\omega)$ is also independent of the choice of ω with probability one, i.e., there are nonrandom sets Σ_{pp}, Σ_{ac} and Σ_{sc}, such that $\sigma_{pp}(H_\omega) = \Sigma_{pp}$, $\sigma_{ac}(H_\omega) = \Sigma_{ac}$, and $\sigma_{sc}(H_\omega) = \Sigma_{sc}$ with probability one. (See [**76, 71, 53, 77, 14, 20**].)

3. Spectral and dynamical localization

Localization can be interpreted from either the spectral or the dynamical point of views. We give selected definitions from each point of view.

By χ_B we denote the characteristic function of the set $B \subset \mathbb{R}^d$ (or $\mathbb{Z}^d$). By χ_x we denote the characteristic function of the cube of side 1 centered at $x \in \mathbb{Z}^d$. We write $\langle x\rangle = \sqrt{1+|x|^2}$. The spectral projection of H_ω is denoted by $E_\omega(\cdot)$. The Hilbert-Schmidt norm of an operator A is written as $\|A\|_2$.