Panoramas & Synthèses
25, 2008, p. 161–213

RANDOM MATRICES AND THE ANDERSON MODEL

by

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Abstract. – In recent years, constructive field techniques and the method of renormalization group around extended singularities have been applied to the weak coupling regime of the Anderson Model. It has allowed to clarify the relationship between this model and the theory of random matrices. We review this situation and the current program to analyze in detail the density of states and Green's functions of this model using the supersymmetric formalism.

***Résumé* (Matrices aléatoires et modèle d'Anderson).** – Ces dernières années, les techniques de champ constructives et la méthode du groupe de renormalisation autour des singularités étendues ont été appliquées au régime à faible couplage du modèle d'Anderson. Cela a permis de clarifier la relation entre ce modèle et la théorie des matrices aléatoires. Nous décrivons cette situation et le programme actuel pour analyser en détail la densité d'états et les fonctions de Green de ce modèle, en utilisant le formalisme supersymétrique.

1. Introduction

This small review is devoted to the elementary theory of random matrices and to the link between this theory and the Anderson model of localization/diffusion of a quantum particle in a random potential.

More precisely we recall first a basic result of random matrix theory, namely the Wigner's semi-circle law for density of states, and give its rigorous derivation through supersymmetric approach.

Then we review the Anderson model, introducing the phase space approach to this model pioneered by Gilles Poirot, and summarizing the results of [**7**].

Finally in the last part we propose some generalizations of the flip random matrix model of [**7**] which are closer to the real Anderson model, using in particular some hierarchical approximations. Though not easy, the control of such more realistic random matrix models seems to be technically feasible and would be an important step towards rigorous theorems about the Anderson model in the weak potential phase.

We thank Abdelmalek Abdesselam, Jean Bellissard, Jacques Magnen and Gilles Poirot for collaboration and discussions on various aspects of this work. V. Rivasseau also thanks F. Klopp for an invitation to lecture on this subject at Paris XIII-Villetanneuse University at the origin of this review.

2. Random Matrices and Wigner's law

Random-matrix theory (RMT) appears in a large number of models both in physics and mathematics, and deals with the statistical properties of large matrices with randomly distributed elements. For a review containing the history and main applications in physics of RMT see [**22**]. For recent developments and applications both in mathematics and physics see [**28**] Ch.1 and the papers in [**2**] (special edition on random matrix theory).

The simplest ensemble of random matrices is the Gaussian unitary ensemble (GUE). It is a probability measure on random hermitian $N \times N$ complex matrices. The coefficients in the upper triangle of the matrix are i.i.d (independent identically distributed) random variables with Gaussian distribution. When the matrix is real symmetric we have the so called Gaussian orthogonal ensemble (GOE). [(1)]. For these ensembles eigenvectors are extended and eigenvalues satisfy Wigner statistics (level-repulsion).

These are also the matrix ensembles most relevant for the connection with Anderson localization. We will see in Section 3 that the discrete Anderson model deals with large real symmetric matrices whose elements have a deterministic and a random part. It seems then logic to expect that the correct ensemble to study is the Gaussian orthogonal ensemble. Nevertheless we will see that after taking the Fourier transform, the relevant matrix is complex self-adjoint, with an additional flip symmetry. The most convenient framework for the problem is then GUE. This does not mean that we expect level-correlations to follow exactly GUE statistics, but more generally Wigner statistics.

2.1. The GUE. – Let H be random hermitian $N \times N$ matrix defined as above. Here the matrix is $H = H_{ij}$, $H = H^*$, hence $H_{ij} = \overline{H}_{ji}$, and

$$P(H) = \frac{1}{Z} \exp\left(-\frac{N}{2} \operatorname{Tr} H^* H\right) \tag{2.1}$$

Z being a normalization factor. The matrix H is made therefore of $N(N-1)/2$ complex variables H_{ij} with $i < j$ and N real ones H_{ii}, so there are N^2 real random variables in H. Since

$$\operatorname{Tr} H^* H = \sum_{i=1}^{N} \sum_{j=1}^{N} |H_{ij}|^2 = \sum_i H_{ii}^2 + 2 \sum_{i<j} (\Re H_{ij}^2 + \Im H_{ij}^2) \tag{2.2}$$

[(1)] Actually the classical ensembles are three, GUE, GOE and GSE according to the invariance properties of the system under time reversal

we have

$$Z = 2^{N/2}(\pi/N)^{N^2/2} \tag{2.3}$$

and the covariance rule is

$$<H_{ij}H^*_{i'j'}> = \frac{1}{N}\delta_{ii'}\delta_{jj'}. \tag{2.4}$$

The scaling factor $\frac{1}{N}$ has been chosen to keep the typical eigenvalues of H of size $O(1)$ as $N \to \infty$; indeed the typical size of the eigenvalues of a random matrix with covariance 1 is obviously of order $\sqrt{N}$, by the law of large numbers.

Physicists would like to know the statistics of the eigenvalues of H, and they are particularly interested in the two first moments of their distribution, called the density of states and the two-level correlation function.

The density of states, $\nu(E)$ for a Hermitian matrix H is the quantity which, when integrated from $-\infty$ to A, counts the number of eigenvalues of H which are lower or equal to A. Since H has exactly N real eigenvalues $\lambda_1, \ldots, \lambda_N$, we have

$$\nu(E) = \frac{1}{N}\operatorname{Tr}\delta(E-H) \tag{2.5}$$

so that $\int_{-\infty}^{+\infty}\nu(E)dE = 1$. We can use the standard formula for the Dirac distribution

$$\delta(x-a) = -\frac{1}{\pi}\lim_{\varepsilon\to 0_+}\Im\frac{1}{x-a+i\varepsilon}. \tag{2.6}$$

Hence

$$\nu(E) = -\frac{1}{\pi N}\lim_{\varepsilon\to 0_+}\Im\operatorname{Tr}\frac{1}{E-H+i\varepsilon}. \tag{2.7}$$

Physicists call $(E-H\pm i\varepsilon)^{-1}$ respectively the retarded and advanced Green's functions for the Hamiltonian H.

The averaged density of states $<\nu(E)>$ is therefore

$$<\nu(E)> = -\lim_{\varepsilon\to 0_+}\int P(H)\,dH\frac{1}{\pi N}\Im\operatorname{Tr}\frac{1}{E-H+i\varepsilon}, \tag{2.8}$$

and $<\nu(E)>dE$ clearly represents the probability for an eigenvalue of H to lie between E and $E+dE$, with normalization condition $\int<\nu(E)>dE = 1$.

The main results on the GUE ensemble is Wigner's semi-circle law:

$$\lim_{N\to\infty}<\nu(E)> = \frac{\chi_{|E|\le 2}}{\pi}\sqrt{1-E^2/4}. \tag{2.9}$$

The corresponding curve is really a semi-ellipse, but of course could be changed into a circle through a slight reparametrization of the covariance of H. The normalization taken here corresponds to $\int E^2\nu(E)dE = 1$.

Wigner's law is a central result. It has been called the non-commutative analog of the Gaussian law of large numbers [**37**], and has been proved to hold in much more general cases than the GUE, for instance for band random matrices [**13**].

The next quantity of interest is the 2-level correlation, which allows to know the conditional probability to find an eigenvalue of H near E knowing already that one

eigenvalue sits at E. More precisely it gives the probability to have two eigenvalues separated by an interval of width ω centered at E, and is therefore

$$R_2(\omega) = \frac{<\nu(E-\omega/2)\nu(E+\omega/2)>}{<\nu(E)>^2}. \tag{2.10}$$

In the GUE, eigenvalues are not independent but tend to "repel" each other. This is seen in the following behavior of the 2-level correlation R_2

$$R_2(s) = \delta(s) + 1 - \frac{\sin^2 \pi s}{(\pi s)^2}, \tag{2.11}$$

where $s = \omega/\Delta$, and Δ is the mean level spacing $\Delta = 1/N<\nu(E)>$. The delta function simply expresses the constraint of presence of an eigenvalue at E. Independence of the eigenvalues would mean $\lim_{s\to 0} R_2(s) - \delta(s) = 1$, hence no change in the probability for a second value to sit near E if a first is present. But here we have $\lim_{s\to 0} R_2(s) - \delta(s) = 0$ because of the $\frac{\sin^2 \pi s}{(\pi s)^2}$ term. Hence there is 0 chance for a second eigenvalue to sit near E if a first one sits at E. This is the phenomenon of "eigenvalue repulsion".

Physicists got intuition of this repulsion by the simple observation of the Vandermonde determinant that appears in the Jacobian of the transformation from the initial coefficients of the matrix to the diagonal eigenvalues and the unitary diagonalizing matrix. In rough terms, we can diagonalize an Hermitian matrix H through a unitary matrix U:

$$H = U\Lambda U^*, \quad U^* = U^{-1}. \tag{2.12}$$

Then one can write the initial measure $P(H)\,dH$ in terms of the coefficients of Λ and U. Clearly the measure on the unitary group must factorize from the eigenvalues measure since $P(H)$ is invariant through action of the unitary group. Let us explain by a simple argument the well known result

$$P(H)\,dH = d\mu(U) e^{-\frac{N}{2}\sum_{i=1}^N \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N d\lambda_i. \tag{2.13}$$

To understand the appearance of the non-trivial Vandermonde factor $\prod_{i<j}(\lambda_i - \lambda_j)^2$ (in addition to the ordinary trivial factor $P(H) = e^{-\frac{N}{2}\sum_{i=1}^N \lambda_i^2}$) we need only to compute the Jacobian at origin from the H variables to the λ variables and the variables parameterizing U near the origin. For this purpose, we can derive the relation $U^*U = 1$ with respect to a set of local parameters U_r for a local chart of the unitary group near the origin. This gives

$$S_r = U^* \frac{\partial U}{\partial U_r} = -S_r^*, \tag{2.14}$$

the tangent space to the unitary group at the origin being the anti-hermitian matrices. From $H = U\Lambda U^*$ one finds

$$\frac{\partial H}{\partial U_r} = \frac{\partial U}{\partial U_r}\Lambda U^* + U\Lambda \frac{\partial U^*}{\partial U_r}, \tag{2.15}$$

hence

$$(U^* \frac{\partial H}{\partial U_r} U)_{ij} = (S_r \Lambda - \Lambda S_r)_{ij} = (S_r)_{ij}(\lambda_j - \lambda_i). \tag{2.16}$$

Furthermore

$$U^* \frac{\partial H_{ij}}{\partial \lambda_k} U = \frac{\partial \Lambda_{ij}}{\partial \lambda_k} = \delta_{ij}\delta_{ik}, \tag{2.17}$$

so that the Jacobian to compute is

$$J = \begin{vmatrix} \frac{\partial H_{ii}}{\partial \lambda_k} & \frac{\partial H_{ij}}{\partial \lambda_k} \\ \frac{\partial H_{ii}}{\partial U_r} & \frac{\partial H_{ij}}{\partial U_r} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ f(\lambda, U) & (\lambda_j - \lambda_i) S_r(U) \end{vmatrix} = g(U) \prod_{i<j} (\lambda_i - \lambda_j)^2. \tag{2.18}$$

Clearly the presence of this Vandermonde determinant means that the eigenvalues of a random matrix in the GUE case are not independent, but repel each other since the measure vanish at coinciding eigenvalues. Physically this level repulsion is analogous to some kind of Pauli exclusion principle between eigenvalues, or to some two body logarithmic interaction:

$$e^{-\frac{N}{2}\sum_{i=1}^N \lambda_i^2} \prod_{i<j} (\lambda_i - \lambda_j)^2 \prod_{i=1}^N d\lambda_i = e^{-\frac{N}{2}\sum_{i=1}^N \lambda_i^2 + 2\sum_{i<j} \log|\lambda_i - \lambda_j|} \prod_{i=1}^N d\lambda_i, \tag{2.19}$$

which is analogous to Coulomb repulsion in two dimensions (also logarithmic).

It is possible to use the theory of orthogonal polynomials to analyze the large N limit and recover Wigner's law for this system or for more complicated non-Gaussian measures on H (for the GUE, orthogonal polynomials are simply Hermite polynomials). This is e.g. done in [**28**]. See also [**30**] for another reference book on the subject.

In this lecture we prefer to stress the supersymmetric approach to this problem. It makes particularly transparent how Wigner's law results from a mean-field theory and a saddle point expansion which expresses the subtle balance between the Gaussian and Vandermonde terms in 2.19.

Supesymmetric approach is basically an algebraic tool that allows to write the averaged density of states (or in general the quantity under study) as a functional integral where a saddle point analysis can be applied. This technique, based on a seminal work by Wegner [**38**] [**34**], was built in a systematic way by Efetov [**15, 16**]. It has proved to be a powerful tool for the study of random systems where classical techniques (such as orthogonal polynomials) do not seem to apply, especially in the context of mesoscopic physics. For a introduction to the method with some applications see [**29**], and also [**20, 41**]. The supersymmetric approach seems also promising for a rigorous analysis (see [**11, 23, 13**]). A detailed study of the GUE density of states in all energy regions and of the finite N corrections can be found in [**12**] (appeared after the submission of this paper).