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CLASSICAL MOTIVES AND MOTIVIC L-FUNCTIONS

by

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The exposition here follows the lecture delivered at the summer school, and hence, contains neither precision, breadth of comprehension, nor depth of insight. The goal rather is the curious one of providing a loose introduction to the excellent introductions that already exist, together with scattered parenthetical commentary. The inadequate nature of the exposition is certainly worst in the third section. As a remedy, the article of Schneider [**39**] is recommended as a good starting point for the complete novice, and that of Nekovář [**36**] might be consulted for more streamlined formalism. For the Bloch-Kato conjectures, the paper of Fontaine and Perrin-Riou [**19**] contains a very systematic treatment, while Kato [**26**] is certainly hard to surpass for inspiration. Kings [**29**], on the other hand, gives a nice summary of results (up to 2003).

1. Motivation

Given a variety X over $\mathbb{Q}$, it is hoped that a suitable analytic function

$$\zeta(X, s),$$

a *ζ-function* of X, encodes important arithmetic invariants of X. The terminology of course stems from the fundamental function

$$\zeta(\mathbb{Q}, s) = \sum_{n=1}^{\infty} n^{-s}$$

named by Riemann, which is interpreted in this general context as the zeta function of Spec($\mathbb{Q}$). A general zeta function should generalize Riemann's function in a manner similar to Dedekind's extension to number fields. Recall that the latter can be defined by replacing the sum over positive integers by a sum over ideals:

$$\zeta(F, s) = \sum_{I} N(I)^{-s}$$

where I runs over the non-zero ideals of the ring of integers $\mathcal{O}_F$ and $N(I) = |\mathcal{O}_F/I|$, and that $\zeta(F,s)$ has a simple pole at $s=1$ (corresponding to the trivial motive factor of Spec(F), as it turns out) with

$$(s-1)\zeta(F,s)|_{s=1} = \frac{2^{r_1}(2\pi)^{r_2}h_F R_F}{w_F\sqrt{|D_F|}}$$

By the unique factorization of ideals, $\zeta(F,s)$ can also be written as an Euler product

$$\prod_{\mathcal{P}}(1-N(\mathcal{P})^{-s})^{-1}$$

as $\mathcal{P}$ runs over the maximal ideals of $\mathcal{O}_F$, that is, the closed points of Spec$(\mathcal{O}_F)$. Now, if a scheme $\mathcal{Y}$ is of finite type over $\mathbb{Z}$, then for any closed point $y \in \mathcal{Y}$, its residue field $k(y)$ is finite. Write $N(y) := |k(y)|$. We can then form an Euler product [**40**]

$$Z(\mathcal{Y},s) := \prod_{y\in\mathcal{Y}_0}(1-N(y)^{-s})^{-1},$$

where $(\cdot)_0$ denotes the set of closed points for any scheme $(\cdot)$. In the case when the map

$$\mathcal{Y} \rightarrow \mathrm{Spec}(\mathbb{Z})$$

factors through Spec$(\mathbb{F}_p)$, $Z(\mathcal{Y},s)$ reduces to Weil's zeta function for a variety over a finite field (with the substitution $p^{-s} \mapsto t$ if a formal variable has intervened as in [**40**], section 1.6).

When we are starting with $X/\mathbb{Q}$, which we assume throughout to be proper and smooth, a straightforward imitation of Dedekind's definition might involve taking an integral model $\mathcal{X}$ of X, which is a proper flat scheme of finite-type over $\mathbb{Z}$ with X as generic fiber, and defining

$$\zeta(X,s)\text{``}:=\text{''}Z(\mathcal{X},s) = \prod_{x\in\mathcal{X}_0}(1-N(x)^{-s})^{-1}$$

The problem with this approach is that the function thus obtained will depend on the model, and there is no general method for choosing a canonical one. However, there will be some set S of primes such that there is a model $\mathcal{X}_S$ over Spec$(\mathbb{Z}[1/S])$ which is furthermore *smooth*. Even though such a $\mathbb{Z}[1/S]$-model need be no more canonical, it does turn out that the incomplete zeta function

$$\zeta_S(X,s) := \prod_{x\in(\mathcal{X}_S)_0}(1-N(x)^{-s})^{-1}$$

is independent of the model. (More on this point below.) So there are good elementary generalizations of incomplete zeta functions. We note in this connection that

$$Z(\mathcal{X},s) = \prod_p Z(\mathcal{X}_p,s)$$

where

$$\mathcal{X}_p = \mathcal{X}\otimes\mathbb{F}_p$$

is the reduction of $\mathcal{X}$ modulo p, so that that

$$\zeta_S(X,s) = \prod_{p\notin S} Z(\mathcal{X}_p, s)$$

is the result of deleting a few Euler factors. Thus, the problem of defining a canonical zeta function becomes one of inserting canonical factors for the primes of bad reduction. It is not impossible that there is a theory of integrals models that isolates a class that is canonical enough to yield a good definition of $\zeta(X,s)$. But the current approach proceeds instead to break up partial zeta functions into natural factors

$$\zeta_S(X,s) = \prod L_S(M_i, s)^{\pm 1},$$

according to the way X is decomposed into constituent motives $\{M_i\}$ in a suitable category. (It is not much of an exaggeration to say that the decomposition of zeta functions is the main empirical phenomenon leading to the hypothesis of a category of motives.) The incomplete L-functions $L_S(M_i,s)$ of the M_i should then encode arithmetic invariants of the M_i, which, in turn, refine the arithmetic invariants of X. It is believed that good analytic properties must be established to access the invariants efficiently, including functional equations. This, in turn, requires us to complete the L-functions using cohomological machinery in general. The completed L-functions then will lead to a completed zeta function.

A simple illustration is provided by the elementary example of an elliptic curve $E/\mathbb{Q}$ with affine equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

Let $\mathcal{E}_S$ be a smooth and proper $\mathbb{Z}[1/S]$ model. Then

$$\zeta_S(E,s) := Z(\mathcal{E}_S, s)$$

It is not very hard to check that

$$\zeta_S(E,s) = \zeta_S(\mathbb{Q},s)\zeta_S(\mathbb{Q},s-1)/L_S(H^1(E),s)$$

([**42**], V.2.4) illustrating the kind of decomposition alluded to above. Here

$$\zeta_S(\mathbb{Q},s) = \sum_{\{(n,p)=1,\forall p\in S\}} n^{-s} = \prod_{p\notin S}(1-p^{-s})^{-1}$$

is a standard incomplete zeta function and

$$L_S(H^1(E),s) = \prod_{p\notin S} L_p(H^1(E),s)$$

is the incomplete L-function of E with factors defined by

$$L_p(H^1(E),s) = \frac{1}{1-a_pp^{-s}+p^{1-2s}}$$

Here $a_p = p+1-N_p$ and N_p is the number of points on E mod p. $L_S(H^1(E),s)$ turns out to be the partial L-function corresponding to the motivic factor $H^1(E)$ of E.

We can put in Euler factors for $p \in S$. It is obvious how to do it for $\zeta_S(\mathbb{Q}, s)$ and $\zeta_S(\mathbb{Q}, s-1)$ giving us the Riemann zeta function $\zeta(\mathbb{Q}, s)$ and its shift $\zeta(\mathbb{Q}, s-1)$ respectively. For the incomplete $L_S(H^1(E), s)$, we put in the factors according to a recipe determined by the reduction of E at p:

$$L_p(H^1(E), s) = \begin{cases} 1/(1-p^{-s}) & \text{split multiplicative;} \\ 1/(1+p^{-s}) & \text{non-split multiplicative;} \\ 1 & \text{additive.} \end{cases}$$

([**43**], II.10) and define

$$L(H^1(E), s) := \prod_p L_p(H^1(E), s)$$

Here we have used the breakdown of the incomplete zeta function into three factors as an aid in defining the full zeta function of E. However, this case is somewhat misleading in that there *is* a canonical model that could have been used instead, namely, the Weierstrass minimal model

$$\mathcal{E}$$

that appears in basic textbooks. In fact, one can check that

$$\zeta(E, s) = \zeta(\mathbb{Q}, s)\zeta(\mathbb{Q}, s-1)/L(H^1(E), s) = Z(\mathcal{E}, s)$$

as follows from the trace formula ([**42**], V.2) for the Frobenius map on elliptic curves for $p \notin S$, and a much easier counting argument for $p \in S$. So this would seem to be an instance where the naive extension of Dedekind's method works out. Nevertheless, we explain how the bad factors can be obtained without reference to the model, starting at this point to use the language of étale cohomology [**34**]. In the sequel, we fix an algebraic closure $\bar{\mathbb{Q}}$ of $\mathbb{Q}$, closures $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and embeddings $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Therefore, we have embeddings of Galois groups

$$G_p := \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow G := \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$$

The residue field of $\bar{\mathbb{Q}}_p$ is an algebraic closure $\bar{\mathbb{F}}_p$ of $\mathbb{F}_p$, and we have an exact sequence

$$0 \to I_p \to G_p \to \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p) \to 0$$

defining the inertia subgroup I_p. Denote by Fr_p the generator of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ that takes x to $x^{1/p}$. Finally, $\bar{\mathcal{E}}_p$ denotes the base-change of $\mathcal{E}_p$ to $\bar{\mathbb{F}}_p$ and $\bar{E}$ the base-change of E to $\bar{\mathbb{Q}}$. We need the étale cohomology

$$H^1(\bar{E}, \mathbb{Q}_l)$$

for primes l, and

$$H^1(\bar{\mathcal{E}}_p, \mathbb{Q}_l)$$

for $l \neq p$. By the Lefschetz trace formula ([**34**], VI.12.3),

$$Z(\mathcal{E}_p, s) = \frac{\det([I - p^{-s}Fr_p]|H^1(\bar{\mathcal{E}}_p, \mathbb{Q}_l))}{\det([I - p^{-s}Fr_p]|H^0(\bar{\mathcal{E}}_p, \mathbb{Q}_l))\det([I - p^{-s}Fr_p]|H^2(\bar{\mathcal{E}}_p, \mathbb{Q}_l))}$$

But for each $i = 0, 1, 2$,

$$H^i(\bar{\mathscr{E}}_p, \mathbb{Q}_l) \simeq H^i(\bar{E}, \mathbb{Q}_l)^{I_p}$$

the superscript referring to the subspace of elements fixed by the inertia action. (For H^0 and H^2, this is an easy exercise. The H^1 case is slightly harder. See [**34**], proof of theorem V.3.5. Although the discussion there is given for smooth surfaces fibered over 'geometric' curves, it is rather straightforward to adapt it to the present situation.) For $p \notin S$, any pair $X \hookrightarrow \mathscr{X}$ as above satisfies

$$H^i(\bar{\mathscr{X}}_p, \mathbb{Q}_l) \simeq H^i(\bar{X}, \mathbb{Q}_l)$$

where the I_p-action must be trivial, and provides the reason that the incomplete zeta function is independent of the model ([**34**], VI.4.1). In any case, it ends up that the bad factor could have been written

$$Z(\mathscr{E}_p, s) = \frac{\det([I - p^{-s}Fr_p]|H^1(\bar{E}, \mathbb{Q}_l)^{I_p})}{\det([I - p^{-s}Fr_p]|H^0(\bar{E}, \mathbb{Q}_l)^{I_p})\det([I - p^{-s}Fr_p]|H^2(\bar{E}, \mathbb{Q}_l)^{I_p})}$$

in a way that refers only to E. It is this formula that generalizes to arbitrary motives.

Since we have thus far been entirely cavalier about convergence, we note in passing that Hasse's bound $|a_p| \leq 2\sqrt{p}$ ([**42**], V.II) implies that the Euler product converges for $Re(s) > 3/2$.

To control fine analytic properties, one establishes a relation to automorphic L-functions. For elliptic curves such a relation can be made explicit by computing the conductor

$$N_E := \prod_{p \in S} p^{f_p}$$

Here

$$f_p = \text{ord}_p(\Delta_E) + 1 - m_E$$

where Δ_E is the discriminant of E and m_E is the number of geometric components (that is, components over $\bar{\mathbb{F}}_p$) of the special fiber of the Neron model of E. Even though this formula for f_p again refers to the model, it can be defined purely in terms of the Galois action on $H^1(\bar{E}, \mathbb{Q}_l)$ ([**43**], IV.10).

The well-known and deep fact, established through the work of Wiles, Taylor-Wiles, and Breuil-Conrad-Diamond-Taylor ([**50**], [**49**], [**9**]), is that L has an analytic continuation to the complex plane. More precisely,

$$L(E, s) = L(f_E, s) = \frac{1}{(2\pi)^s\Gamma(s)} \int_0^\infty f_E(iy)y^{s-1}dy$$

$$= \frac{1}{(2\pi)^s\Gamma(s)}[\int_{1/\sqrt{N_E}}^\infty f_E(iy)y^{s-1}dy + w_E \int_{1/\sqrt{N_E}}^\infty f_E(iy)y^{1-s}dy]$$

for a normalized weight 2 new cusp form f_E of level N_E which is an eigenvector for the Hecke operators, determined by a q-expansion

$$f_E = q + a_2q^2 + \cdots$$

where the a_p have to be the same as those for E and the general coefficient is determined by those with prime index. The number $w_E = \pm 1$ in this formula is an intrinsic