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# LECTURES ON SHIMURA VARIETIES

*by*

Alain Genestier & Bao Châu Ngô

***Abstract.*** – The main goal of these lectures is to explain the representability of the moduli space of abelian varieties with polarizations, endomorphisms and level structures, due to Mumford [**13**], and the description of the set of its points over a finite field, due to Kottwitz [**9**]. We also try to motivate the general definition of Shimura varieties and their canonical models as in the article of Deligne [**5**]. We will leave aside important topics like compactifications, bad reduction and the $p$-adic uniformization of Shimura varieties.

This is the notes of the lectures on Shimura varieties delivered by one of us in the Asia-French summer school organized at IHÉS in July 2006. It is basically based on the notes of a course delivered by the two of us in Université Paris-Nord in 2002.

***Résumé.*** – Le but principal de cet exposé est d'expliquer la représentabilité des espaces de modules de variétés abéliennes à travers les polarisations, endomorphismes et structures de niveau, due à Mumford [**13**] et la description de l'ensemble de ses points au-dessus d'un corps fini, due à Kottwitz [**9**]. Nous essayons également de motiver la définition générale des variétés de Shimura et de leurs modèles canoniques selon l'article de Deligne [**5**]. Nous laissons de côté des sujets importants comme les compactifications, la mauvaise réduction et l'uniformisation $p$-adique des variétés de Shimura.

Il d'agit de notes d'exposés sur les variétés de Shimura, donnés par l'un des auteurs lors de l'École d'été franco-asiatique organisée à l'IHÉS en juillet 2006. Ce texte est basé sur les notes d'un cours délivré par l'un des auteurs à l'université Paris-Nord en 2002.

## 1. Quotients of Siegel's upper half space

**1.1. Review on complex tori and abelian varieties.** – Let $V$ denote a complex vector space of dimension $n$ and $U$ a lattice in $V$ which is by definition a discrete subgroup of $V$ of rank $2n$. The quotient $X = V/U$ of $V$ by $U$ acting on $V$ by translation, is naturally equipped with a structure of compact complex manifold and a structure of abelian group.

***Lemma 1.1.1**. – We have canonical isomorphisms from* $\mathrm{H}^r(X,\mathbb{Z})$ *to the group of alternating* $r$*-forms* $\bigwedge^r U \to \mathbb{Z}$.

*Proof.* Since $X = V/U$ with $V$ contractible, $\mathrm{H}^1(X,\mathbb{Z}) = \mathrm{Hom}(U,\mathbb{Z})$. The cup-product defines a homomorphism

$$\bigwedge^r \mathrm{H}^1(X,\mathbb{Z}) \to \mathrm{H}^r(X,\mathbb{Z})$$

which is an isomorphism since $X$ is isomorphic to $(S_1)^{2n}$ as real manifolds (where $S_1 = \mathbb{R}/\mathbb{Z}$ is the unit circle). □

Let $L$ be a holomorphic line bundle over the compact complex variety $X$. Its Chern class $c_1(L) \in \mathrm{H}^2(X,\mathbb{Z})$ is an alternating 2-form on $U$ which can be made explicit as follows. By pulling back $L$ to $V$ by the quotient morphism $\pi : V \to X$, we get a trivial line bundle since every holomorphic line bundle over a complex vector space is trivial. We choose an isomorphism $\pi^*L \to \mathcal{O}_V$. For every $u \in U$, the canonical isomorphism $u^*\pi^*L \simeq \pi^*L$ gives rise to an automorphism of $\mathcal{O}_V$ which is given by an invertible holomorphic function

$$e_u \in \Gamma(V, \mathcal{O}_V^\times).$$

The collection of these invertible holomorphic functions for all $u \in U$, satisfies the cocycle equation

$$e_{u+u'}(z) = e_u(z+u')e_{u'}(z).$$

If we write $e_u(z) = e^{2\pi i f_u(z)}$ where $f_u(z)$ are holomorphic function well defined up to a constant in $\mathbb{Z}$, the above cocycle equation is equivalent to

$$F(u_1,u_2) = f_{u_2}(z+u_1) + f_{u_1}(z) - f_{u_1+u_2}(z) \in \mathbb{Z}.$$

The Chern class

$$c_1 : \mathrm{H}^1(X,\mathcal{O}_X^\times) \to \mathrm{H}^2(X,\mathbb{Z})$$

sends the class of $L$ in $\mathrm{H}^1(X,\mathcal{O}_X^\times)$ on the element $c_1(L) \in \mathrm{H}^2(X,\mathbb{Z})$ whose the corresponding 2-form $E : \bigwedge^2 U \to \mathbb{Z}$ is given by

$$(u_1,u_2) \mapsto E(u_1,u_2) := F(u_1,u_2) - F(u_2,u_1).$$

***Lemma 1.1.2**. – The Neron-Severi group* $\mathrm{NS}(X)$*, defined as the image of* $c_1 : \mathrm{H}^1(X,\mathcal{O}_X^\times) \to \mathrm{H}^2(X,\mathbb{Z})$ *consists of the alternating 2-form* $E : \bigwedge^2 U \to \mathbb{Z}$ *satisfying the equation*

$$E(iu_1, iu_2) = E(u_1,u_2)\,,$$

*where* $E$ *still denotes the alternating 2-form extended to* $U \otimes_{\mathbb{Z}} \mathbb{R} = V$ *by* $\mathbb{R}$*-linearity.*

*Proof.* The short exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X^\times \to \mathcal{O}_X \to 0$$

induces a long exact sequence which contains

$$\mathrm{H}^1(X,\mathcal{O}_X^\times) \to \mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathcal{O}_X).$$

It follows that the Neron-Severi group is the kernel of the map

$$\mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathscr{O}_X).$$

This map is the composition of the obvious maps

$$\mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathbb{C}) \to \mathrm{H}^2(X,\mathscr{O}_X).$$

The Hodge decomposition

$$\mathrm{H}^m(X,\mathbb{C}) = \bigoplus_{p+q=m} \mathrm{H}^p(X,\Omega^q_X)$$

where $\Omega^q_X$ is the sheaf of holomorphic $q$-forms on $X$, can be made explicit [**15**, page 4]. For $m=1$, we have

$$\mathrm{H}^1(X,\mathbb{C}) = V^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = V^*_{\mathbb{C}} \oplus \overline{V}^*_{\mathbb{C}},$$

where $V^*_{\mathbb{C}}$ is the space of $\mathbb{C}$-linear maps $V \to \mathbb{C}$, $V^*_{\mathbb{C}}$ is the space of conjugate $\mathbb{C}$-linear maps and $V^*_{\mathbb{R}}$ is the space of $\mathbb{R}$-linear maps $V \to \mathbb{R}$. There is a canonical isomorphism $\mathrm{H}^0(X,\Omega^1_X) = V^*_{\mathbb{C}}$ defined by evaluating a holomorphic 1-form on $X$ on the tangent space $V$ of $X$ at the origin. There is also a canonical isomorphism $\mathrm{H}^1(X,\mathscr{O}_X) = \overline{V}^*_{\mathbb{C}}$.

By taking $\bigwedge^2$ on both sides, the Hodge decomposition of $\mathrm{H}^2(X,\mathbb{C})$ can also be made explicit. We have $\mathrm{H}^2(X,\mathscr{O}_X) = \bigwedge^2 \overline{V}^*_{\mathbb{C}}$, $\mathrm{H}^1(X,\Omega^1_X) = V^*_{\mathbb{C}} \otimes \overline{V}^*_{\mathbb{C}}$ and $\mathrm{H}^0(X,\Omega^2_X) = \bigwedge^2 V^*_{\mathbb{C}}$. It follows that the map $\mathrm{H}^2(X,\mathbb{Z}) \to \mathrm{H}^2(X,\mathscr{O}_X)$ is the obvious map $\bigwedge^2 U^*_{\mathbb{Z}} \to \bigwedge^2 V^*_{\mathbb{C}}$. Its kernel is precisely the set of integral 2-forms $E$ on $U$ which satisfy the relation $E(iu_1,iu_2) = E(u_1,u_2)$ (when they are extended to $V$ by $\mathbb{R}$-linearity). □

Let $E : \bigwedge^2 U \to \mathbb{Z}$ be an integral alternating 2-form on $U$ satisfying $E(iu_1,iu_2) = E(u_1,u_2)$ after extension to $V$ by $\mathbb{R}$-linearity. The real 2-form $E$ on $V$ defines a Hermitian form $\lambda$ on the $\mathbb{C}$-vector space $V$ by

$$\lambda(x,y) = E(ix,y) + iE(x,y)$$

which in turn determines $E$ by the relation $E = \mathrm{Im}(\lambda)$. The Neron-Severi group $\mathrm{NS}(X)$ can be described in yet another way as the group of the Hermitian forms $\lambda$ on the $\mathbb{C}$-vector space $V$ having an imaginary part which takes integral values on $U$.

***Theorem 1.1.3* (Appell-Humbert).** – *Isomorphy classes of holomorphic line bundles on $X = V/U$ correspond bijectively to pairs $(\lambda,\alpha)$, where*

– *$\lambda \in NS(X)$ is an Hermitian form on $V$ such that its imaginary part takes integral values on $U$*
– *$\alpha : U \to S_1$ is a map from $U$ to the unit circle $S_1$ satisfying the equation*

$$\alpha(u_1+u_2) = e^{i\pi \mathrm{Im}(\lambda)(u_1,u_2)}\alpha(u_1)\alpha(u_2).$$

*For every $(\lambda,\alpha)$ as above, the line bundle $L(\lambda,\alpha)$ is given by the Appell-Humbert cocycle*

$$e_u(z) = \alpha(u)e^{\pi\lambda(z,u)+\frac{1}{2}\pi\lambda(u,u)}.$$

Let $\mathrm{Pic}(X)$ be the abelian group consisting of the isomorphy classes of line bundles on $X$ and $\mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$ be the kernel of the Chern class. We have an exact sequence :

$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \to \mathrm{NS}(X) \to 0.$$

Let us also write: $\hat{X} = \mathrm{Pic}^0(X)$; it is the group consisting of characters $\alpha : U \to S_1$ from $U$ to the unit circle $S_1$. Let $V_{\mathbb{R}}^* = \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R})$. There is a homomorphism $V_{\mathbb{R}}^* \to \hat{X}$ sending $v^* \in V_{\mathbb{R}}^*$ to the line bundle $L(0, \alpha)$, where $\alpha : U \to S_1$ is the character

$$\alpha(u) = \exp(2i\pi\langle u, v^*\rangle).$$

This induces an isomorphism $V_{\mathbb{R}}^*/U^* \to \hat{X}$, where

$$U^* = \{u^* \in V_{\mathbb{R}}^* \text{ such that } \forall u \in U,\ \langle u, u^*\rangle \in \mathbb{Z}\}.$$

Let us denote by the semi-linear dual, consisting of $\mathbb{C}$-semi-linear maps $V \to \mathbb{C}$. We can identify $\overline{V}_{\mathbb{C}}^*$ with the $\mathbb{R}$-dual $V_{\mathbb{R}}^*$ by the $\mathbb{R}$-linear bijection sending a semi-linear $f$ to its imaginary part $g$ ($f$ can be recovered from $g$: use the formula $f(v) = -g(iv) + ig(v)$). This gives $\hat{X} = \overline{V}_{\mathbb{C}}^*/U^*$ a structure of complex torus; it is called *the dual complex torus of* $X$. With respect to this complex structure, the universal line bundle over $X \times \hat{X}$ given by Appell-Humbert formula is a holomorphic line bundle.

A Hermitian form on $V$ induces a $\mathbb{C}$-linear map $V \to \overline{V}_{\mathbb{C}}^*$. If moreover its imaginary part takes integral values in $U$, the linear map $V \to \overline{V}_{\mathbb{C}}^*$ takes $U$ into $U^*$ and therefore induces a homomorphism $\lambda : X \to \hat{X}$ which is symmetric (i.e. such that $\hat{\lambda} = \lambda$ with respect to the obvious identification $X \simeq \hat{\hat{X}}$). In this way, we identify the Neron-Severi group $\mathrm{NS}(X)$ with the group of symmetric homomorphisms from $X$ to $\hat{X}$.

Let $(\lambda, \alpha)$ be as in the theorem and let $\theta \in \mathrm{H}^0(X, L(\lambda, \alpha))$ be a global section of $L(\lambda, \alpha)$. Pulled back to $V$, $\theta$ becomes a holomorphic function on $V$ which satisfies the equation

$$\theta(z + u) = e_u(z)\theta(z) = \alpha(u)e^{\pi\lambda(z,u) + \frac{1}{2}\pi\lambda(u,u)}\theta(z).$$

Such a function is called a theta-function with respect to the hermitian form $\lambda$ and the multiplier $\alpha$. The Hermitian form $\lambda$ needs to be positive definite for $L(\lambda, \alpha)$ to have a lot of sections, see [**15**, § 3].

***Theorem 1.1.4***. – *The line bundle $L(\lambda, \alpha)$ is ample if and only if the Hermitian form $H$ is positive definite. In that case,*

$$\dim \mathrm{H}^0(X, L(\lambda, \alpha)) = \sqrt{\det(E)}.$$

Consider the case where $H$ is degenerate. Let $W$ be the kernel of $H$ or of $E$, i.e.

$$W = \{x \in V | E(x, y) = 0, \forall y \in V\}.$$

Since $E$ is integral on $U \times U$, $W \cap U$ is a lattice of $W$. In particular, $W/W \cap U$ is compact. For any $x \in X$, $u \in W \cap U$, we have

$$|\theta(x + u)| = |\theta(x)|$$

for all $d \in \mathbb{N}$, $\theta \in \mathrm{H}^0(X, L(\lambda,\alpha)^{\otimes d})$. By the maximum principle, it follows that $\theta$ is constant on the cosets of $X$ modulo $W$ and therefore $L(\lambda,\alpha)$ is not ample. Similar argument shows that if $H$ is not positive definite, $L(H,\alpha)$ can not be ample, see [**15**, p.26].

If the Hermitian form $H$ is positive definite, then the equality

$$\dim \mathrm{H}^0(X, L(\lambda,\alpha)) = \sqrt{\det(E)}$$

holds. In [**15**, p.27], Mumford shows how to construct a basis, well-defined up to a scalar, of the vector space $\mathrm{H}^0(X, L(\lambda,\alpha))$ after choosing a sublattice $U' \subset U$ of rank $n$ which is Lagrangian with respect to the symplectic form $E$ and such that $U' = U \cap \mathbb{R}U'$. Based on the equality $\dim \mathrm{H}^0(X, L(\lambda,\alpha)^{\otimes d}) = d^n\sqrt{\det(E)}$, one can prove $L(\lambda,\alpha)^{\otimes 3}$ gives rise to a projective embedding of $X$ for any positive definite Hermitian form $\lambda$. See Theorem 2.2.3 for a more complete statement. $\square$

***Definition 1.1.5***. – 1. *An abelian variety is a complex torus that can be embedded into a projective space.*
2. *A polarization of an abelian variety $X = V/U$ is an alternating form $\lambda : \bigwedge^2 U \to \mathbb{Z}$ which is the Chern class of an ample line bundle.*

With a suitable choice of a basis of $U$, $\lambda$ can be represented by a matrix

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix},$$

where $D$ is a diagonal matrix $D = (d_1, \dots, d_n)$ for some non-negative integers $d_1, \dots, d_n$ such that $d_1|d_2|\dots|d_n$. The form $E$ is non-degenerate if these integers are nonzero. We call $D = (d_1, \dots, d_n)$ the *type of the polarization* $\lambda$. A polarization is called *principal* if its type is $(1, \dots, 1)$.

***Corollary 1.1.6* (Riemann)**. – *A complex torus $X = V/U$ can be embedded as a closed complex submanifold into a projective space if and only if there exists a positive definite hermitian form $\lambda$ on $V$ such that the restriction of its imaginary part to $U$ is a (symplectic) 2-form with integral values.*

Let us rewrite Riemann's theorem in term of matrices. We choose a $\mathbb{C}$-basis $e_1, \dots, e_n$ for $V$ and a $\mathbb{Z}$-basis $u_1, \dots, u_{2n}$ of $U$. Let $\Pi$ be the $n \times 2n$-matrix $\Pi = (\lambda_{ji})$ with $u_i = \sum_{j=1}^n \lambda_{ji} e_j$ for all $i = 1, \dots, 2n$. $\Pi$ is called *the period matrix.* Since $\lambda_1, \dots, \lambda_{2n}$ form an $\mathbb{R}$-basis of $V$, the $2n \times 2n$-matrix $\begin{pmatrix} \Pi \\ \overline{\Pi} \end{pmatrix}$ is invertible. The alternating form $E : \bigwedge^2 U \to \mathbb{Z}$ is represented by an alternating matrix, also denoted by $E$, with respect to the $\mathbb{Z}$-basis $u_1, \dots, u_{2n}$. The form $\lambda : V \times V \to \mathbb{C}$ given by $\lambda(x,y) = E(ix,y) + iE(x,y)$ is hermitian if and only if $\Pi E^{-1}\,{}^t\Pi = 0$. When this condition is satisfied, the Hermitian form $\lambda$ is positive definite if and only if the symmetric matrix $i\Pi E^{-1}\,{}^t\overline{\Pi}$ is positive definite.