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MOTIVES FROM A CATEGORICAL POINT OF VIEW

by

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This article is based on the second overview lecture given at the Workshop. The principal aim of these survey lectures was to provide a bird's eye view of the theory of motives vis-a-vis some of the longer courses and special lectures that were to follow. Needless to say, such a sweeping overview involves compressing a vast area, thereby necessitating omission of many details. In this article, we have largely retained the flavor of the lecture, introducing various concepts, themes and conjectures from the theory of motives.

Apart from the references in the bibliography, the interested reader is also referred to the various articles on this subject in the homepages of B. Kahn, M. Levine and J. Milne.

1. Introduction

Classical Galois theory relates finite groups to the study of polynomial equations over fields. The theory of Motivic Galois groups is a vast higher dimensional analogue, wherein 'motives' are related to finite dimensional representations of some groups, called the 'Motivic Galois groups'. The study of motives encompasses deep questions coming from such diverse areas as Hodge theory, algebraic cycles, arithmetic geometry and Galois representations.

The essential idea is the following. If G is any group, and F a field, then the category $\mathbf{Rep}_F(G)$ of finite dimensional F-representations of G has a rich structure, namely that of a 'Tannakian category'. Consider the association

$$\begin{array}{ccc} \text{Groups} & \longrightarrow & \text{Tannakian categories} \\ G & \mapsto & \mathbf{Rep}_F(G). \end{array}$$

If G is compact, then the classical theorem of Tannaka and Krein shows how the group G may be recovered from its category of representations via the obvious forgetful functor

$$\mathbf{Rep}_F(G) \longrightarrow \mathbf{Vec}_F$$

into the category $\mathbf{Vec}_F$ of finite dimensional F-vector spaces.

An analogous theorem for algebraic groups or group schemes will be discussed below. The idea is to first attach group schemes to various suitable Tannakian (sub)categories arising from the theory of motives, using realizations (Betti, Hodge, l-adic....). The group schemes associated to these categories have the property that their corresponding categories of representations are in fact equivalent to the original categories that we started with. Schematically,

$$\begin{array}{c} \text{Category } \mathscr{M} \text{ coming from motives} \\ \wr \\ \downarrow \\ G' \quad \text{pro} - \text{algebraic group schemes} \\ \wr \\ \downarrow \\ \mathbf{Rep}\ G' \longleftrightarrow \mathscr{M}, \end{array}$$

where the first vertical arrow denotes the association mentioned above and the last two sided arrow denotes equivalence. This is made more precise in the language of Tannakian categories.

2. Tannakian Categories

The main references for this section are [**10, 27**] and the article by L. Breen in [**2**], Part 1.

Definition 2.1. – Let R be a commutative ring. An *R-linear category* is a category $\mathscr{C}$ such that for every pair of objects M, N in $\mathscr{C}$, the set of morphisms $\mathscr{C}(M,N)$ is an R-module, and the composition law is R-bilinear. In addition, we shall also need that finite sums exist in$\mathscr{C}$. An R-functor between two such categories is an R-linear functor.

We impose additional conditions on such an R-linear category in the definition below, referring the reader to any of the references mentioned above, for more details.

Definition 2.2. – A *tensor category* over R is an R-linear category $\mathscr{C}$ with an R-bilinear tensor functor $\otimes$,

$$\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$$

which satisfies the commutativity and associativity constraints, and such that there exists a unit object $\mathbb{I}$ in $\mathscr{C}$.

In particular, given objects L, M and N there are the following functorial isomorphisms in a tensor category:

$$\begin{array}{ll} \alpha_{LMN}: & L\otimes(M\otimes N)\simeq(L\otimes M)\otimes N \\ c_{MN}: & M\otimes N\simeq N\otimes M, \text{ with } c_{MN}\circ c_{NM}=1_{M\otimes N} \\ u_M: & M\otimes\mathbb{I}\simeq M,\ u'_M:\mathbb{I}\otimes M\simeq M, \end{array}$$

such that various compatibilities are expressed by the natural commutative diagrams. We remark that other equivalent terminology for a tensor category is $\otimes$-category ACU (Saavedra-Rivano) or symmetric monoidal category.

Definition 2.3. – The category $\mathscr{C}$ has an *internal hom* functor

$$\begin{array}{rcl} \mathrm{hom}:\mathscr{C}\times\mathscr{C} & \to & \mathscr{C} \\ (X,Y) & \mapsto & \mathrm{hom}(X,Y) \end{array}$$

if $\mathrm{hom}(X,Y)$ is the representing object for the functor

$$\begin{array}{rcl} \mathscr{C}^{\mathrm{op}} & \to & \mathbf{Sets} \\ M & \mapsto & \mathscr{C}(M\otimes X,Y). \end{array}$$

Suppose that the internal hom functor exists in $\mathscr{C}$. Then the *dual object* $M^\vee$ for every object M of $\mathscr{C}$ is defined by $M^\vee=\mathrm{hom}(M,\mathbb{I})$.

We thus have a duality functor

$$\begin{array}{rcl} \vee:\mathscr{C} & \longrightarrow & \mathscr{C}^{\mathrm{op}} \\ M & \mapsto & M^\vee \\ \{M\overset{f}{\mapsto}N\} & \mapsto & \{N^\vee\overset{{}^tf}{\to}M^\vee\} \end{array}$$

and evaluation maps for every $M\in\mathscr{C}$,

$$\text{(evaluation)}\quad \varepsilon:\quad M\otimes M^\vee\ \to\ \mathbb{I}.$$

The category is said to be *rigid* if there are also *coevaluation maps* η for every object $M\in\mathscr{C}$,

$$\text{(coevaluation)}\quad \eta:\quad \mathbb{I}\ \to\ M^\vee\otimes M$$

with the property that the composites below

$$M\overset{u_M^{-1}}{\simeq}M\otimes\mathbb{I}\overset{1_M\otimes\eta}{\longrightarrow}M\otimes M^\vee\otimes M\overset{\varepsilon\otimes 1_M}{\longrightarrow}\mathbb{I}\otimes M\simeq M,$$

$$M^\vee\overset{(u'_{M^\vee})^{-1}}{\simeq}\mathbb{I}\otimes M^\vee\overset{\eta\otimes 1_M}{\longrightarrow}M^\vee\otimes M\otimes M^\vee\overset{1_M\otimes\varepsilon}{\longrightarrow}M^\vee\otimes\mathbb{I}\overset{u_{M^\vee}}{\simeq}M^\vee,$$

are respectively 1_M and $1_{M^\vee}$. Further, there are functorial isomorphisms

$$\mathrm{hom}(Z,\mathrm{hom}(X,Y))\simeq\mathrm{hom}(Z\otimes X,Y),\qquad X^\vee\otimes Y\simeq\mathrm{hom}(X,Y).$$

Definition 2.4. – Given an R-rigid tensor category $\mathscr{C}$, every endomorphism $f \in \mathrm{End}(M)$ has a *trace*, denoted by $\mathrm{tr}(f)$, which is an element of the commutative R-algebra $\mathrm{End}(\mathbb{I})$. It is defined as the composite

$$\mathbb{I} \xrightarrow{\eta} M^\vee \otimes M \xrightarrow{1_{M^\vee}\otimes f} M^\vee \otimes M \xrightarrow{c_{M^\vee M}} M \otimes M^\vee \xrightarrow{\varepsilon} \mathbb{I}.$$

We thus get a map

$$\mathrm{tr} : \mathrm{End}(M) \to \mathrm{End}(\mathbb{I})$$

for every object M of $\mathscr{C}$. The *dimension* or *rank* of an object M in $\mathscr{C}$ is then defined as

$$\dim M := \mathrm{tr}(I_M)$$

Examples. (1) The prototype is $\mathscr{C} = \mathbf{Rep}_F(G)$ where F is a field and G any group. The usual tensor product of representations gives the tensor functor while $\mathbb{I}$ is the trivial representation and $\vee$ denotes the contragredient representation functor. The notions of trace and dimension are the usual ones. More generally, if R is a commutative ring, the category of R-modules is a rigid tensor category.

(2) Let F be a field, and $\mathscr{C} := \mathbf{VecGr}_F$ be the category of $\mathbb{Z}$-graded F-vector spaces (V_n) such that $\underset{n}{\oplus} V_n$ has finite dimension. We shall mainly consider this category, but with the Koszul rule for the commutativity constraint. In other words, consider the isomorphisms

$$\overset{\star}{\phi} : V \otimes W \simeq W \otimes V,$$

with $\overset{\star}{\phi} = \underset{r,s}{\oplus} (-1)^{rs} \phi^{r,s}$, where

$$\phi^{r,s} : V^r \otimes W^s \to W^s \otimes V^r$$

is the usual isomorphism in $\mathscr{C}$. With this latter definition, if $V = (V_n)$ is an object of $\mathscr{C}$, then the rank of V is the 'super-dimension' $\dim V^+ - \dim V^-$, where $V^+ = \oplus V^{2k}$ and $V^- = \oplus V^{2k+1}$. With the usual tensor functor $\otimes_F$, the category $\mathscr{C}$ is a rigid tensor category.

(3) The category of vector bundles over a variety X/F is a F-rigid tensor category.

A tensor functor $\Phi : \mathscr{C} \to \mathscr{C}'$ between tensor categories is a functor preserving the tensor structure, i.e. there exist functorial isomorphisms

$$\kappa_{M,N} : \Phi(M) \otimes \Phi(N) \simeq \Phi(M \otimes N)$$

which are compatible with the associativity and commutativity constraints and such that the identity object of $\mathscr{C}$ is mapped to that of $\mathscr{C}'$. If further, $\mathscr{C}$ and $\mathscr{C}'$ are rigid, then there are functorial isomorphisms

$$\Phi(M^\vee) \simeq \Phi(M)^\vee.$$

There is an obvious notion of tensor equivalence between tensor categories. Further, we have $\mathrm{tr}(\Phi(f)) = \Phi(\mathrm{tr}(f))$ and $\dim(\Phi(M)) = \Phi(\dim(M))$.

If Φ and Φ' are tensor functors, then $\hom^{\otimes}(\Phi, \Phi')$ is the set of morphisms (i.e. natural transformations) of tensor functors. Further, if the categories $\mathscr{C}$ and $\mathscr{C}'$ are rigid, then any morphism of tensor functors is an isomorphism.

We now outline how the set $\hom^{\otimes}$ is given an additional structure. For any field F and an F-algebra R, there is a canonical $\otimes$-functor,

$$\begin{array}{rcl}\Phi_R : \mathbf{Vec}_F & \to & \underline{\mathrm{Mod}}_R \\ V & \mapsto & V \otimes_F R.\end{array}$$

If Ψ and Λ are tensor functors from $\mathscr{C} \to \mathbf{Vec}_F$, then we define $\hom^{\otimes}(\Psi, \Lambda)$ to be the functor from the category of F-algebras to the category of sets such that

$$\hom^{\otimes}(\Psi, \Lambda)(R) = \hom^{\otimes}(\Phi_R \circ \Psi, \Phi_R \circ \Lambda).$$

Definition 2.5. – An *additive (resp. abelian) tensor category* is a tensor category $\mathscr{C}$ over R such that $\mathscr{C}$ is additive (resp. abelian) and the tensor functor is biadditive. If in addition, we have $R = \mathrm{End}(\mathbb{I})$, then such a category is said to be an *additive* (resp. *abelian*) *tensorial* category.

There is the notion of tensor subcategories generated by subsets of objects; briefly this is the smallest tensor subcategory containing the generating set of objects.

We now come to the important notion of fibre functors which is crucial to define Tannakian categories.

Definition 2.6. – Let $R = F$ be a field and $\mathscr{C}$ a rigid abelian tensorial category so that $\mathrm{End}(\mathbb{I}) = F$. A *fibre functor* on $\mathscr{C}$ is a faithful, exact, tensor functor

$$\omega : \mathscr{C} \to \mathbf{Vec}_{F'}$$

into the rigid category of F'-vector spaces of finite dimension over F', where F' is an unspecified algebraic field extension of F.

We say that $\mathscr{C}$ is *Tannakian* if $\mathscr{C}$ has a fibre functor; $\mathscr{C}$ is *neutral* if $\mathscr{C}$ has a fibre functor into $\mathbf{Vec}_F$ and $\mathscr{C}$ is *neutralized* if a fibre functor into $\mathbf{Vec}_F$ has been specified.

Given a fibre functor $\omega : \mathscr{C} \to \mathbf{Vec}_F$, one can define the affine group scheme $\mathbf{G}(\omega)$ over F by

$$\mathbf{G}(\omega) = \mathrm{Aut}^{\otimes} \omega$$

where the latter is viewed as a scheme via its 'functor of points' on F-algebras.

The following deep theorem is the centerpiece of Tannakian formalism.