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A PLURALITY OF (NON)VISUALIZATIONS: BRANCH POINTS AND BRANCH CURVES AT THE TURN OF THE 19TH CENTURY

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Abstract. — This article deals with the different ways branch points and branch curves were visualized at the turn of the 19th century. On the one hand, for branch points of complex curves one finds an abundance of visualization techniques employed. German mathematicians such as Felix Klein or Walther von Dyck were the main promoters of these numerous forms of visualization, which appeared either as two-dimensional illustrations or three-dimensional material models. This plurality of visualization techniques, however, also resulted in inadequate images that aimed to show the varied ways branch points could possibly be represented. For branch (and ramification) curves of complex surfaces, on the other hand, there were hardly any representations. When the Italian school of algebraic geometry studied branch curves systematically only partial illustrations can be seen, and branch curves were generally made "invisible". The plurality of visualizations shifted into various forms of non-visualization. This can be seen in the different ways visualization techniques disappeared.

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RÉSUMÉ (Une diversité de visualisations et non-visualisations : points de branchement et courbes de ramification autour de 1900.)

L'article traite des différentes façons de visualiser les points et les courbes de branchement autour de 1900. De nombreuses techniques de visualisation ont été employées pour les points de branchement de courbes complexes. Des mathématiciens allemands comme Felix Klein ou Walther von Dyck ont été les principaux promoteurs de cette multitude de visualisations, que ce soit sous la forme d'illustrations ou de modèles matériels tridimensionnels. Cependant, cette pluralité de techniques a également été à l'origine d'images inadéquates visant à montrer les diverses manières possibles de représenter des points de branchement. Pour les courbes de branchement (et de ramification) de surfaces complexes, il est difficile de trouver une visualisation. Lorsque les courbes de branchement ont été systématiquement étudiées par l'école italienne de géométrie algébrique, seules des illustrations partielles ont pu être trouvées, et les courbes de branchement ont été généralement rendues « invisibles ». La pluralité des visualisations s'est transformée en une pluralité de non-visualisations, dont témoignent différents modes de disparition des techniques de visualisation.

INTRODUCTION

Ever since Bernhard Riemann (1826–1866) introduced the now well known *Riemann surfaces* in his 1851 doctoral dissertation on complex function theory, as the covering of the complex line (or of the projective complex line) for multi-valued analytic functions in a complex region, attempts have been made to visualize these coverings—and especially their branch points. The question concerning how to visualize these functions was also dealt with before Riemann's introduction of curves as covering: a complex valued curve $y = f(x)$ is embedded in a four dimensional space $\mathbb{C}^2$; every point (x_0, y_0), when $x_0, y_0 \in \mathbb{C}$ such that $y_0 = f(x_0)$ can be represented then in a four-dimensional real space $\mathbb{R}^4$ via a quadruple $(\mathrm{Re}(x_0), \mathrm{Im}(x_0), \mathrm{Re}(y_0), \mathrm{Im}(y_0))$. Hence visualizing these complex points (x_0, y_0) as a drawing on paper (by drawing for example only the *real* points in $\mathbb{R}^2$, i.e., the points for which $\mathrm{Im}(x_0) = \mathrm{Im}(y_0) = 0$) or as model in a three-dimensional space (by constructing models of surfaces whose points are either $(\mathrm{Re}(x_0), \mathrm{Im}(x_0), \mathrm{Im}(y_0))$ or $(\mathrm{Re}(x_0), \mathrm{Im}(x_0), \mathrm{Re}(y_0))$ would always risk being insufficient from a mathematical as well as from a visual point of view. [1] Notwithstanding this insufficiency, Riemann's con-

[1] To recall: for a given complex number $c = a + bi$, (where $i = \sqrt{-1}$), $\mathrm{Re}(c) = a$, $\mathrm{Im}(c) = b$.

cept of the complex curve as a covering, as I will show, prompted a variety of visualizations.

The present article deals with the various visualizations of a special phenomenon arising when considering these curves as covering of the complex line. To give an example, consider the function $y^2 = x - 1$ and its projection to the x-axis:

$$p : \{(x, y) \in \mathbb{C}^2 : y^2 = x - 1\} \to \mathbb{C}, (x, y) \mapsto x.$$

Generically, every point $x' \in \mathbb{C}$ has two different preimages (x', y_1), $(x', y_2) \in \mathbb{C}^2$ such that $(y_1)^2 = x' - 1$ and $(y_2)^2 = x' - 1$. However, for $x' = 1$, the number of the preimages is less than two (explicitly, there is only one preimage: $(1, 0)$). One might say that when considering the points $x'' \in \mathbb{C}$ which are close to $x' = 1$, the two preimages of x'' "come together," or "coincide" into one point when x'' approaches x'. Considering only *smooth* functions, these points, whose number of preimages is lower than the expected one, are called *branch points*;[2] while the points on the curve, for which few of the preimages "come together," are called—in current terminology—*ramification points*. However, as the terminology regarding these points was not standardized in the 19th century, they were also usually referred to as branch points ("Verzweigungspunkte" or "Windungspunkte" in German), a usage I will follow. It should also be noted that when n preimages "come together," one says that the branch point is of order $n - 1$.

The same phenomenon may also happen when considering complex surfaces as a cover of the complex plane $\mathbb{C}^2$, when in this situation, the collection of branch points is in fact a complex curve in $\mathbb{C}^2$, called the *branch curve* of the complex surface (when considered as a covering).[3] The question that this paper would like to answer concerns the nature of the various visualizations of branch points and branch curves during the 19th and the 20th century. More precisely, the paper, concentrat-

2 In fact, the map p can be any surjective holomorphic map between a Riemann surface and the projective complex line (using current terminology).

3 And the corresponding curve on the surface is called in current terminology *ramification curve* (see Section II). The explicit computation of branch curves (and also of branch points) can be easily done—from a computational point of view—, at least when one deals with projections. For example, given a cubic surface: $f(z) = z^3 - 3az + b$, where a and b are homogeneous forms in (x, y, w) of degrees 2 and 3 respectively, and the projection is given by $(x, y, w, z) \to (x, y, w)$. In these coordinates, the ramification curve is given by the intersection of the surface and its derivative with respect to z, i.e., of $f = 0$ and $df/dz = 0 = z^2 - a = 0$ and the branch curve B is therefore given by $b^2 - 4a^3 = 0$, being a curve of degree 6.

ing on the years between 1874 and 1929, aims to show in Section I that while for *branch points* (of finite order)—either on the complex line or on the curve—there was an abundance of visualizations or a plurality of visual interpretations, for *branch curves*, the situation, as I will examine in Section II, was actually quite the opposite: while for branch points the different three-dimensional models and two-dimensional illustrations were at times epistemological and stimulated further research, for branch curves, similar illustrations—in the cases when they even existed—were mostly considered technically;[4] visualization techniques were ignored or considered unnecessary. It is here where one notices a shift in the mathematical practice of visualization: from a plurality of such techniques to either a rejection of them or partial visualization of an "auxiliary machinery," not, however, of the object itself. In some cases, this "auxiliary machinery" eventually became the object of research itself. In most cases, however, as will be elaborated in the concluding Section III, one can see that the plurality of visualizations was replaced by a plurality of non-visualizations, prompted by different modes of disappearance.

1. BRANCH POINTS: EPISTEMOLOGICAL VISUALIZATIONS

In this first section, I will deal with what may be thought of as a counter position to the situation concerning the visualization of branch curves, a topic that will be dealt with in the second section. This section will aim to show how branch points of complex curves were usually thought of during the second half of the 19th century as what could (and should) be visualized. This does not mean that all of the attempts at visualizing them

[4] With these distinctions I follow throughout this article Hans-Jörg Rheinberger's differentiation between epistemic and technical objects. According to Rheinberger "epistemic objects [...] present themselves in a characteristic, irreducible vagueness. This vagueness is inevitable because, paradoxically, epistemic things embody what one does not yet know." [Rheinberger 1997, p. 28] These objects, their purpose, or the field of research that they open and the questions that they may propose are not yet defined or not yet canonically categorized. This is exactly what makes them into an epistemological object, as they are in the process of becoming "well-defined" or "stable." But "in contrast to epistemic objects, [...] experimental conditions"—and technical objects, as Rheinberger later adds—"tend to be characteristically determined within the given standards of purity and precision. [...] they restrict and constrain" the scientific objects [Rheinberger 1997, p. 29]. But while it seems that there is a clear distinction between the not yet defined epistemological object and the clearly defined technical one, Rheinberger immediately adds "The difference between experimental conditions and epistemic things, therefore, is functional rather than structural." [Rheinberger 1997, p. 30]

were considered successful, satisfactory or even accepted by the entire mathematical community. What I aim to show, by contrast, is how these attempts were directed at illustrating and showing what branch points *looked like*. Given that the research on the history of Riemann surfaces is vast, a full-blown examination of how branch points were visualized during this period and afterwards is beyond the scope of this paper.[5] Thus, for example, I will not deal with Hermann Weyl's influential book *Die Idee der Riemannschen Fläche* [Weyl 1913]. Rather I will examine a few different examples, especially from the last quarter of the 19th century and the first quarter of the 20th century, which indicate that the research of branch points of Riemann surfaces was coupled not only with analytical investigation within the domain of function theory, or with algebraic calculations, but also with visual practices.

1.1. *1850–1865: Puiseux, Riemann and Neumann*

A year before Riemann's presentation of his dissertation, Victor Puiseux (1820–1883) in 1850 published his manuscript *Recherches sur les fonctions algébriques*, dealing with complex functions defined by an equation $f(u,z) = 0$. Puiseux, one might say, viewed complex curves as a covering of the complex line $\mathbb{C}$, which would be defined, as noted above, using contemporary notation, as follows:

$$\mathrm{pr} : \{(u,z) \in \mathbb{C}^2 : f(u,z) = 0\} \to \mathbb{C}, (u,z) \mapsto z.$$

Assuming that for the function $f(u,z)$ the degree of z is p, given a complex point z_0 on the z-axis, Puiseux asks what would happen to the p solutions of the equation $f(u,z_0) = 0$, that is, to the points in the set $\mathrm{pr}^{-1}(z_0) = u_1(z_0), \ldots, u_p(z_0)$, when the point z_0 moves along a closed path, which avoids passing through points z' for which two or more values $u_i(z')$ coincide (recall that the z axis is the complex line $\mathbb{C}$, which is topo-

[5] On the development of the concept of the n-dimensional manifold beginning from the 1850s with Riemann and his concept of covering, see e.g., [Scholz 1980; 1999] The topic concerning the various visualizations of Riemann surfaces (and not necessarily their branch points), starting from the second half of the 19th century, also deserves a more elaborate discussion than that presented here, one which would also take into account their digital visualization.

For an extensive survey of Riemann's work and the responses to it, see [Gray 2015, p. 153–194]; see also [Bottazzini & Gray 2013, p. 259–341] for a similar discussion, also containing other figures of branch points, similar to what is shown in this paper; Bottazzini and Gray show that Gustav Holzmüller in his 1882 book *Einführung in die Theorie der isogonalen Verwandtschaften und der conformen Abbildungen* [Holzmüller 1882, p. 271] and Felice Casorati with his sketches of Riemann surfaces in 1864 also drew figures of different visualizations of branch points [Neuenschwander 1998, p. 23].