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Tobias JÄGER & Daniel LENZ & Christian OERTEL

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Annales Scientifiques de l'École Normale Supérieure,
45, rue d'Ulm, 75230 Paris Cedex 05, France.
Tél. : (33) 1 44 32 20 88. Fax : (33) 1 44 32 20 80.
`annales@ens.fr`

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Société Mathématique de France
Case 916 - Luminy
13288 Marseille Cedex 09
Tél. : (33) 04 91 26 74 64
Fax : (33) 04 91 41 17 51
email : `abonnements@smf.emath.fr`

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MODEL SETS WITH POSITIVE ENTROPY IN EUCLIDEAN CUT AND PROJECT SCHEMES

BY Tobias JÄGER, Daniel LENZ AND Christian OERTEL

ABSTRACT. – We construct model sets arising from cut and project schemes in Euclidean spaces whose associated Delone dynamical systems have positive topological entropy. The construction works both with windows that are proper and with windows that have empty interior. In a probabilistic construction with randomly generated windows, the entropy almost surely turns out to be proportional to the measure of the boundary of the window.

RÉSUMÉ. – On construit des ensembles de Delone euclidiens obtenus par coupe et projection de sorte que l'entropie des systèmes dynamiques associés soit strictement positive. La construction permet d'utiliser une fenêtre propre ou d'intérieur vide. Dans une construction probabiliste, pour presque tout paramètre, l'entropie est proportionnelle à la mesure de la frontière de la fenêtre.

1. Introduction

In the last decades, aperiodic order—often referred to as the mathematical theory of quasicrystals—has developed into a broad and highly active field of research, see e.g., [2, 16] for recent books dealing with this topic. In this context, the main attention has been given to models with a strong degree of long-range order. In particular, there is nowadays a fairly good understanding of the relations between pure point diffraction – characterizing quasicrystals from the physical viewpoint—and purely discrete dynamical spectrum, which has emerged as one of the major tools in the mathematical analysis of long-range aperiodic order.

In this paper, we have a slightly different focus and construct models that may be considered as intermediate between strong long-range order and disorder. More precisely, we introduce a broad family of model sets, produced by cut and project schemes in Euclidean space, whose associated Delone dynamical systems exhibit a high degree of chaoticity, including positive topological entropy. At the same time, they still inherit a certain degree

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of long-range order, which is built into the underlying cut and project scheme and manifests itself in a non-vanishing discrete part of the dynamical spectrum as well as in minimality. Although we restrict here to study the basic dynamical properties, we hope that the constructed models may be instrumental in understanding the transition from quasicrystalline to amorphous configurations in solid matter. We note several recent works dealing with similar model sets with 'thick boundary' of the window, based on a variety of different methods [3, 4, 26, 14, 17]. The reader may take that as an indication for the timeliness of the endeavor.

We will discuss more specifically how the present paper relates to other works and contributes to the emerging general theory towards the end of this section, after we have introduced the necessary notation. Here, we already note that - to the best of our knowledge - it provides the first examples of model sets with positive entropy based on Euclidean cut and project schemes.

A *cut and project scheme (CPS)* is a triple $(G, H, \mathcal{L})$ consisting of locally compact abelian groups G, called *direct space*, and H, called *internal space*, and a discrete co-compact subgroup (*lattice*) $\mathcal{L} \subseteq G \times H$ such that the canonical projection $\pi_G : G \times H \to G$ is one-to-one and the canonical projection $\pi_H : G \times H \to H$ has dense image. This framework goes back to Meyer's influential book [22] and has later been developed in [23, 24, 28]. In this paper we will always take $G = \mathbb{R}^N$ and we will assume H to be σ-compact (i.e., a countable union of compact sets) and metrizable. Our main application concerns the case $G = H = \mathbb{R}$. So, the reader may also well think from the very beginning of H as just another Euclidean space $\mathbb{R}^M$ (where $M \neq N$ is possible).

Given a relatively compact subset $W \subseteq H$, which is called a *window* in this context, such a CPS produces a uniformly discrete subset of G via

$$⅄(W) = \pi_G\left(\mathcal{L} \cap (G \times W)\right).$$

An alternative way to define $⅄(W)$ is to introduce the *star-map*. Set $L := \pi_G(\mathcal{L})$ and $L^* := \pi_H(\mathcal{L})$. Then, the star map $* : L \to L^*$ is given by $\ell \mapsto \ell^*$, where ℓ^* is uniquely defined by $(\ell, \ell^*) \in \mathcal{L}$ due to the injectivity of $\pi_{G|\mathcal{L}}$. Then, we have

$$⅄(W) = \{\ell \in L \mid \ell^* \in W\}.$$

If W has non-empty interior, then $⅄(W)$ is called a *model set*, in the general case it is called a *weak model set*. We will be concerned with model sets whose window has a further 'smoothness' feature: A window $W \subseteq H$ is called *proper* (or sometimes *topologically regular*) if

$$\mathrm{cl}(\mathrm{int}(W)) = W.$$

The associated model set will then also be referred to as *proper model set*. Note that any proper window is compact.

A model set is always Delone set (see the next section for more detailed definitions and a discussion of further facts concerning CPS and model sets).

Given a window $W \subseteq H$ (which will mostly be compact in our considerations below), we can associate a dynamical system to $⅄(W)$ by considering the $\mathbb{R}^N$-action $(s, \Lambda) \mapsto \Lambda - s$ on the *hull* of $⅄(W)$. This hull is given as $\Omega(⅄(W)) = \mathrm{cl}(\{⅄(W) - s \mid s \in \mathbb{R}^N\}$, where the closure is taken in a suitable topology (defined below). The properties of this dynamical system depend crucially on the boundary of the window W.

If W is proper and the boundary of W has Haar measure zero, then the dynamical system $(\Omega(\curlywedge(W)), \mathbb{R}^N)$ is (measurably) isomorphic to the Kronecker flow on the torus $\mathbb{T} = (\mathbb{R}^N \times H)/\mathcal{L}$ defined by $\omega : \mathbb{R}^N \times \mathbb{T} \to \mathbb{T}$, $(s, \xi) \mapsto \xi + [s, 0]_{\mathcal{L}}$ and is therefore uniquely ergodic with purely discrete dynamical spectrum [28] and zero topological entropy [7]. This case has attracted most attention in recent years. In fact, it seems fair to say that *regular model sets*, i.e., sets of the form $\curlywedge(W)$ for proper W whose boundary has measure zero, are the prime examples for quasicrystals. In particular, substantial efforts have been spent over the years to prove pure point diffraction for regular model sets, see e.g., [12, 28]. By now this pure pointedness is well understood and three different approaches have been developed: the approach of [12] via Poisson summation formula has recently been extended to a very general framework in [27]. The result of [28] can be seen within the context of the equivalence between purely discrete dynamical spectrum and pure point diffraction, proven in this setting in [19] and later generalized in various directions in e.g., [5, 11, 21, 20]. Finally, pure point spectrum can also be shown using almost periodicity [8], see also [29].

Conversely, the case of windows with 'thick boundary', in the sense of positive Haar measure, is not as well understood. A general idea in this context is that thickness of the boundary should imply positive topological entropy and failure of unique ergodicity. In fact, corresponding conjectures have been brought forward by Moody, see [14] for discussion, and Schlottmann [28]. These conjectures are supported by prominent examples. Indeed, for the well-known example of visible lattice points the associated dynamical system is far from being uniquely ergodic and has positive topological entropy [9, 26]. This system has still pure point diffraction [9] and pure point dynamical spectrum if it is equipped with a natural ergodic measure [13]. Existence of such a canonical ergodic measure for general model sets with thick boundary has received attention recently, see [3] for an approach based on a maximal density condition and [17] for a rather structural approach. Quite remarkably, all these model sets with maximal density still have pure point diffraction and pure point dynamical spectrum with respect to the canonical measure [3]. Note, however, that the eigenfunctions will in general not be continuous anymore. In this context, a general upper bound on topological entropy has been established in [14]. Given this support for the mentioned conjectures, the recent findings in [4] may seem surprising as they provide examples of proper model sets with thick boundary which are still uniquely ergodic (and minimal) with topological entropy zero. At the same time [4] also provides some examples of proper model sets with minimal dynamical systems of positive entropy lacking unique ergodicity. All examples of [4] are based on Toeplitz systems.

In all examples in the preceding discussion, where the topological entropy was shown to be positive, the internal space H is not an Euclidean space but has a rather more complicated structure (being a p-adic space in the case of the visible lattice points and being an odometer in the case of the Toeplitz systems). In the present paper we provide examples of model sets with positive entropy based on Euclidean internal space.

For the sake of simplicity, we will here restrict to Euclidean CPS with one-dimensional internal space $H = \mathbb{R}$. In principle, similar constructions can be carried out with higher-dimensional internal group, see Section 8 for a brief discussion. Then, a lattice with the above